Data-based Control for Linear Time-invariant Discrete-time Systems

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Abstract: This paper proposes a new framework for control system design, called the data-based control approach or data space approach, in which the input and output data of a dynamical system is directly and solely used to analyze or design a control system without the employment of any mathematical models like transfer functions, state space equations, and kernel representations. Since, in this approach, most of the analysis and design processes are carried out in the domain of the data space, we introduce some notions of geometrical objects, e.g., the open-loop and closed-loop data spaces, which serve as the system representations in the data space. In addition, we establish a relationship between the open-loop and closed-loop data spaces that the closed-loop data space is contained in the open-loop data space as one of its subspaces. By using this relationship, we can derive the data-based stabilization condition for a linear time-invariant discrete-time system, which leads to a linear matrix inequality with a rank constraint.

Keywords: data-based control, data space approach, input-output data

1. Introduction

In the field of control theory, it is widely accepted that the model-based control method, which employs the mathematical models such as transfer functions, state space equations, and kernel representations as a system representation, is the most effective and reliable approach for control system analysis and synthesis. Particularly, in this approach, the mathematical models may be obtained in two main different ways: either by the analytical formulation from the first principles, or by the identification from the input and output data.

From here, we briefly mention about the basic motivation of the present study as follows: When only the input and output data is available as reliable information about a dynamical system – which is much more common case in practice – then it is not inevitable to use the mathematical models for analysis and synthesis of control systems. That is, it is possible that there exists another approach as an alternative for the model-based control approach.

In particular, the authors are convinced that there exists a new framework, in which we can analyze and design control systems with the direct use of the input and output data without taking the pains to identify the mathematical models from it. In this framework, we also believe that it is possible to deal more directly with the uncertainties in data, which is mainly caused by noise, compared with the model-based control methods.

This viewpoint was first proposed by the second author and his colleagues, and several preliminary results on the model-less algorithm for tracking control based on input-output data have been introduced in [1], [2]. In addition, the data-based stability conditions for open-loop and closed-loop systems, respectively, have been proposed by the authors in [3], which enable us to check the internal stability of a linear time-invariant discrete-time system directly from its behavior, namely, the input and output data.

In the present paper, we propose a new framework for feedback control design based on the input and output data, which is called the *data-based control approach* or *data space approach*. In addition, we establish the data-based stabilization condition for a linear time-invariant discrete-time system.

In this approach, we deal extensively with the geometrical objects such as the open-loop and closed-loop data spaces, to which all the open-loop and closed-loop behaviors of the system respectively belong. Hence, in Section 2, we introduce the notions of these two data spaces, which serve as the system representations in the data space. In addition, in the data space approach, a feedback control can be realized by imposing an additional constraint on the open-loop data space, which results in a closed-loop data space. Hence, we also introduce the notion of the feedback data space which will act as the additional constraint. From these notions of data spaces, we can establish a relationship between the open-loop and closed-loop data spaces that the closed-loop data space is contained in the open-loop data space as one of its subspaces.

By using this relationship between the open-loop and closed-loop data spaces and the data-based stability condition for closed-loop systems proposed in [3], we can derive the data-based stabilization condition, which leads to a linear matrix inequality (LMI) with a rank constraint. However, the rank constraint makes the data-based stabilization problem non-convex and NP-hard. In this paper, an approximation algorithm by using the LMI relaxation and linearization method [6], [7] is applied to solve the data-based stabilization problem. In simulations, the feasibility of the proposed data-based stabilization condition is demonstrated through two simple numerical examples.

Throughout the paper, all the discussions are made upon the following assumptions:

- The plant under consideration is a finite-dimensional linear time-invariant system, and its order and relative degree are known *a priori*.
- The input and output data is properly sampled and noise-free.

2. Data Spaces

In this section, we introduce the notions of the geometrical objects such as the open-loop, closed-loop, and feedback data spaces, which serve as the representations of the system and controller in the data space approach.

2.1. Open-loop data space

Let us consider a linear time-invariant discrete-time system with single input and single output as the plant and describe its dynamics by using an input and output difference equation as

$$\begin{aligned} & y(k+n)+a_{n-1}y(k+n-1)+\cdots+a_1y(k+1)+a_0y(k) \\ & = b_mu(k+m)+\cdots+b_1u(k+1)+b_0u(k), \quad k\in\mathbb{N}, \end{aligned} \tag{1}$$

where $u(k)\in\mathbb{R}$ and $y(k)\in\mathbb{R}$ denote the input and output data at the time instant k, respectively. In addition, the order of the system and its relative degree are given as n and $n-m$ $(n>m)$ respectively, thus a_0 and b_0 are not zero at the same time and b_m is nonzero as well.

In the model-based control approach, the dynamics of the system (1) is modeled in terms of a parameter vector $\theta^T\in\mathbb{R}^{n+m+2}$ as

$$\theta=(1,a_{n-1},\cdots,a_1,a_0,-b_m,\cdots,-b_1,-b_0), \tag{2}$$

which can be identified as a point in the parameter space of $\mathbb{R}^{n+m+2}$.

On the other hand, in the data-based control approach, we are concerned only with the input and output data of (1), not with the parameter vector θ as in (2). From this viewpoint, we introduce a set of data vectors $d(k)\in\mathbb{R}^{n+m+2}$, each of which consists of both $(n+1)$ consecutive output data and $(m+1)$ consecutive input data of (1), defined as

$$d(k)=\left(y(k+n),\cdots,y(k),u(k+m),\cdots,u(k)\right)^T, \quad k\in\mathbb{N}. \tag{3}$$

Here, let $\mathcal{D}$ denote the data space of $\mathbb{R}^{n+m+2}$. Then, due to the constraint in (1), the degrees of freedom of the data vectors $d(k)$ are reduced by 1, hence the set of data vectors $d(k)$ in (3) represent a certain subspace, i.e., $(n+m+1)$-dimensional hyperplane, in the data space $\mathcal{D}$. Therefore, as a representation of the system (1), we can define a subspace of $\mathcal{D}$, which is called the *open-loop data space*, as follows:

$$\mathcal{D}_o\triangleq\{d(k)\in\mathcal{D}\mid d(k)\in\theta^\perp \ \ \forall u(k),\, k\in\mathbb{N}\}, \tag{4}$$

where $\dim\mathcal{D}_o=n+m+1$.

In addition, let us consider a set of data matrix $\Psi_o\in\mathbb{R}^{(n+m+2)\times(n+m+1)}$ whose column vectors consist of $(n+m+1)$ consecutive data vectors of (3) as follows:

$$\Psi_o(k)\triangleq\begin{bmatrix}d(k) & d(k+1) & \cdots & d(k+n+m)\end{bmatrix}, \quad k\in\mathbb{N}. \tag{5}$$

If it holds that $\operatorname{span}\Psi_o(k)=\mathcal{D}_o$ for some instant k, then we call the data matrix $\Psi_o(k)$ as a basis matrix of the open-loop data space $\mathcal{D}_o$ and denote it as Ψ_o.

2.2. Feedback data space

For the stabilization of the system (1), we consider a feedback control in which all the available input and output data in (1) are fed back to generate the control input. However, in this approach, we view the feedback control as an operation that imposes an additional constraint on the original open-loop data space, which results in a closed-loop data space.

Therefore, from this consideration, one of such feedback control can be realized by constructing a linear time-invariant dynamic controller given as

$$\begin{aligned} & u(k+m)+d_{m-1}u(k+m-1)+\cdots+d_1u(k+1)+d_0u(k) \\ & = c_{n-1}y(k+n-1)+\cdots+c_1y(k+1)+c_0y(k), \quad k\in\mathbb{N}, \end{aligned} \tag{6}$$

where m and $m-n\,(m<n)$ are given as the order of the controller and its relative degree respectively.

Note that the controller in (6) is not proper, which essentially means that the output data $y(k+m),\cdots,y(k+n-1)$ in (6) are not yet available from the measurements when we attempt to feedback them to determine the control input $u(k+m)$ at each instant k. However, since these output data are already determined by the system dynamics and the inputs given prior to the $(k+m)$ step, it is possible to predict them from the past input and output data, although the details of which are not given in the present paper. As a result, the controller in (6) is causal and hence can be realized by feeding back both the measured and predicted output data with the past input data to calculate the control input data $u(k+m)$, $k\in\mathbb{N}$.

From (6), we can also define a subspace of $\mathcal{D}$, which is called the *feedback data space*, as

$$\mathcal{D}_f\triangleq\{d(k)\in\mathcal{D}\mid d(k)\in\theta_c^\perp,\ k\in\mathbb{N}\}, \tag{7}$$

where $\dim\mathcal{D}_f=n+m+1$ and

$$\theta_c=(0,-c_{n-1},\cdots,-c_1,-c_0,1,d_{m-1},\cdots,d_1,d_0). \tag{8}$$

Here, the parameter vector $\theta_c^T\in\mathbb{R}^{n+m+2}$ in (8) itself represents the feedback gain. To find this feedback gain is the ultimate goal of the data-based control approach, similarly with the model-based control approach.

2.3. Closed-loop data space

In the data space approach, the geometrical interconnection of two data spaces, i.e., the open-loop and feedback data space, yields the dynamics of closed-loop systems.

As in the case of the open-loop data space, let us consider a set of data vectors $\hat{d}(k)\in\mathcal{D}$, each of which consists of the closed-loop input and output data, defined as

$$\hat{d}(k)=\left(\hat{y}(k+n),\cdots,\hat{y}(k),\hat{u}(k+m),\cdots,\hat{u}(k)\right)^T, \ k\in\mathbb{N}, \tag{9}$$

where $\hat{u}(k)$ and $\hat{y}(k)$ denote the closed-loop input and output data that satisfy (1) and (6) simultaneously.

Then, it can be easily seen that all the closed-loop data vectors $\hat{d}(k)$ in (9) belong to a certain subspace of $\mathcal{D}$ that is the intersection of the open-loop data space $\mathcal{D}_o$ and feedback data space $\mathcal{D}_f$. Hence, we refer to this subspace as the *closed-loop data space* and define it as follows:

$$\mathcal{D}_c\triangleq\{\hat{d}(k)\in\mathcal{D}\mid \hat{d}(k)\in\theta^\perp \ \text{ and } \ \hat{d}(k)\in\theta_c^\perp,\ \ k\in\mathbb{N}\}, \tag{10}$$

where $\dim\mathcal{D}_c=n+m$.

In addition, we also consider a closed-loop data matrix $\Psi_c\in\mathbb{R}^{(n+m+2)\times(n+m)}$ whose column vectors consist of $(n+m)$ consecutive closed-loop data vectors of (9) as follows:

$$\Psi_c(k)\triangleq\begin{bmatrix}\hat{d}(k) & \hat{d}(k+1) & \cdots & \hat{d}(k+n+m-1)\end{bmatrix}, \ k\in\mathbb{N}. \tag{11}$$

For some instant k, if we have the data matrix $\Psi_c(k)$ such that $\operatorname{span}\Psi_c(k)=\mathcal{D}_c$, then we call it a basis matrix of $\mathcal{D}_c$ and denote it as Ψ_c.

3. Feedback in Data Space

In this section, we investigate the geometrical meaning of a relationship that exists between the open-loop and closed-loop data spaces. This relationship is established by a feedback operation in the data space, by which all the behaviors in the open-loop data space are converted into those in the closed-loop data space.

$$
\left[\begin{array}{ccc:c:ccc}
1 & & & 0 & & & \\
& \ddots & & \vdots & & [0] & \\
& & 1 & 0 & & & \\
\hdashline
0 & \cdots & 0 & 0 & 1 & 0 \cdots & 0 \\
\hdashline
& & & 0 & 1 & & \\
& [0] & & \vdots & & \ddots & \\
& & & 0 & & & 1
\end{array}\right]
\left(\begin{array}{c}
\hat{y}(k+n) \\ \vdots \\ \hat{y}(k+2) \\ \hat{y}(k+1) \\ \hdashline \hat{u}(k+m+1) \\ \hdashline \hat{u}(k+m) \\ \vdots \\ \hat{u}(k+2) \\ \hat{u}(k+1)
\end{array}\right)
=
\left[\begin{array}{cccc:c:cccc}
-a_{n-1} & \cdots & -a_1 & -a_0 & b_m & b_{m-1} & \cdots & b_1 & b_0 \\
1 & & & 0 & 0 & & & & \\
& \ddots & & \vdots & \vdots & & [0] & & \\
& & 1 & 0 & 0 & & & & \\
\hdashline
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\hdashline
c_{n-1} & \cdots & c_1 & c_0 & 0 & -d_{m-1} & \cdots & -d_1 & -d_0 \\
& & & & 0 & 1 & & & 0 \\
& [0] & & & \vdots & & \ddots & & \vdots \\
& & & & 0 & & & 1 & 0
\end{array}\right]
\left(\begin{array}{c}
\hat{y}(k+n-1) \\ \vdots \\ \hat{y}(k+1) \\ \hat{y}(k) \\ \hdashline \hat{u}(k+m) \\ \hdashline \hat{u}(k+m-1) \\ \vdots \\ \hat{u}(k+1) \\ \hat{u}(k)
\end{array}\right)
$$

$$
\triangleq \; E\,x_c(k+1) = A_c\,x_c(k), \quad k \in \mathbb{N}, \qquad E, A_c \in \mathbb{R}^{(n+m+1)\times(n+m+1)}. \tag{14}
$$

First, it can be easily seen that the definition of the closed-loop data space $\mathcal{D}_c$ in (10) itself reveals the fact that the closed-loop data space $\mathcal{D}_c$ is always contained in the open-loop data space $\mathcal{D}_o$ as one of its subspaces as

$$
\mathcal{D}_c \subset \mathcal{D}_o, \tag{12}
$$

which is clear from $\mathcal{D}_c = \mathcal{D}_o \cap \mathcal{D}_f$.

From this, we have the following result that specifies the relationship between the open-loop and closed-loop data spaces:

Theorem 1. *Let a basis matrix Ψ_o of $\mathcal{D}_o$ and a basis matrix Ψ_c of $\mathcal{D}_c$ be given, then there exists a coefficient matrix $Z \in \mathbb{R}^{(n+m+1)\times(n+m)}$ with the full column rank such that*

$$
\Psi_c = \Psi_o Z. \tag{13}
$$

Proof: It is trivial from (12). □

Remark 1. The relationship in (13) can be interpreted that the basis of the closed-loop data space $\mathcal{D}_c$ is made up of a subset of the basis of the open-loop data space $\mathcal{D}_o$, which is selected by the coefficient matrix Z. In addition, the relationship in (13) will be used in deriving the data-based stabilization condition in Section 5.

4. Data-based Stability Condition for Closed-loop Systems

In this section, we introduce the data-based stability condition for closed-loop systems, which is originally intended to be used as a stabilization condition in the data-based control approach.

From (1) and (6), we can construct the entire closed-loop system in a descriptor representation as seen in (14), where the vector $x_c \in \mathbb{R}^{n+m+1}$ corresponds to both n consecutive closed-loop output data and $(m+1)$ consecutive closed-loop input data. In particular, the entire modes of (14) consist of the n-th order dynamic modes of the plant (1), the m-th order dynamic modes of the controller (6), and the static constraint of a feedback coupling. In addition, it can be easily seen that the system (14) is regular and causal from [4]

$$
\begin{aligned}
\det(zE - A_c) &= z^{n+m} + (a_{n-1} + d_{m-1} - b_m c_{n-1}) z^{n+m-1} \\
&\quad + \cdots + (a_0 d_0 - b_0 c_0) \not\equiv 0, \qquad (15) \\
\deg(\det(zE - A_c)) &= \operatorname{rank} E.
\end{aligned}
$$

Here, we also define the data matrix $\Phi_c \in \mathbb{R}^{(n+m+1)\times(n+m)}$ which consists of $(n+m)$ consecutive vectors x_c as follows:

$$
\Phi_c(k) = \begin{bmatrix} x_c(k) & x_c(k+1) & \cdots & x_c(k+n+m-1) \end{bmatrix}, \quad k \in \mathbb{N}, \tag{16}
$$

by which we have another representation of (14) as

$$
E\,\Phi_c(k+1) = A_c\,\Phi_c(k), \quad k \in \mathbb{N}. \tag{17}
$$

From these settings, let us introduce the data-based stability condition for closed-loop systems as follows:

Theorem 2. *When the system* (14) *is regular and causal, the following statements are equivalent.*

1) The system (14) *has an asymptotically stable equilibrium point.*

2) $\exists P = P^T > 0$ s.t.
$x_c^T(k+1)(E^T P E)x_c(k+1) - x_c^T(k)(E^T P E)x_c(k) < 0$
$\forall x_c(k) \neq 0,\ k \in \mathbb{N}$ s.t. $E x_c(k) \neq 0$ and $E x_c(k+1) = A_c x_c(k)$.

3) $\exists P = P^T > 0\ \exists k_0$ s.t.

$$
\Psi_c^T(k_0) \left[\begin{array}{c:c} E & 0 \\ \hdashline 0 & E \end{array}\right]^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \left[\begin{array}{c:c} E & 0 \\ \hdashline 0 & E \end{array}\right] \Psi_c(k_0) < 0
$$

for $\Psi_c(k_0)$ s.t. $\operatorname{rank} \Psi_c(k_0) = n+m$.

Proof: The proof is given in Appendix A. □

Remark 2. In Theorem 2, the condition 3) represents itself the data-based stability condition for closed-loop systems, in which no information relating to the system matrix A_c is employed except the data matrix $\Psi_c(k_0)$ of rank $(n+m)$. Note also that the matrix E is only related to and uniquely determined by the order and relative degree of the system and controller.

However, for a closed-loop system, if the initial conditions and feedback gains of the controller are not given in advance, we can not generate the closed-loop input and output data as seen in $\Psi_c(k_0)$. This means that the data matrix $\Psi_c(k_0)$ in the condition 3) of Theorem 2 describes the variable matrix that has to be decided. Therefore, when the data-based stability condition in Theorem 2 is associated with some open-loop input and output data, it can be reformulated as a data-based stabilization condition, and the details will be discussed in the next section.

5. Data-based Stabilization

In this section, by using the relationship between the open-loop and closed-loop data spaces in Theorem 1 and the data-based stability condition for closed-loop systems as seen in the condition 3) of Theorem 2, we derive the data-based stabilization condition for a linear time-invariant discrete-time system. In addition, we will show how the feedback gain can be determined from the solution of the data-based stabilization problem.

5.1. Data-based stabilization condition

By simply substituting $\Psi_c(k_0)$ in the condition 3) of Theorem 2 with $\Psi_{\text{o}} Z$ as seen in (13), we obtain the data-based stabilization condition as follows:

Find $P, \;\; Z$ s.t.

$$\bullet P = P^T > 0 \tag{18}$$

$$\bullet Z^T \Psi_{\text{o}}^T \left[\begin{array}{c|c} E & 0 \\ \hline 0 & E \end{array}\right]^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \left[\begin{array}{c|c} E & 0 \\ \hline 0 & E \end{array}\right] \Psi_{\text{o}} Z < 0, \tag{19}$$

where (19) describes a nonlinear matrix inequality. Unfortunately, it seems to be extremely difficult to obtain a solution of the data-based stabilization problem in (18) and (19) since the nonlinear matrix inequality in (19) makes the problem non-convex and NP-hard.

From now on, we convert the original data-based stabilization condition in (18) and (19) into a computationally more tractable form. First, by Finsler's theorem [5], the following equivalent condition can be easily obtained:

Find $P, \;\; \mu, \;\; W$ s.t.

$$\bullet P = P^T > 0, \;\; \mu > 0 \tag{20}$$

$$\bullet \Psi_{\text{o}}^T \left[\begin{array}{c|c} E & 0 \\ \hline 0 & E \end{array}\right]^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \left[\begin{array}{c|c} E & 0 \\ \hline 0 & E \end{array}\right] \Psi_{\text{o}} < \mu(W^T W), \tag{21}$$

where $\mu \in \mathbb{R}$, and $W^T \in \mathbb{R}^{n+m+1}$ denotes a basis for the nullspace of Z as

$$WZ = 0. \tag{22}$$

From (21), since both μ and W represent the variable matrices, by adopting $Q \in \mathbb{R}^{(n+m+1)\times(n+m+1)}$ as a new variable matrix as

$$Q = \mu(W^T W) \geq 0, \tag{23}$$

we can derive an equivalent data-based stabilization condition as follows:

Definition 1 (Data-based Stabilization Problem). *For a given basis matrix Ψ_{o} of the open-loop data space $\mathcal{D}_{\text{o}}$, find a Lyapunov matrix P and a matrix Q such that*

$$\bullet \; P = P^T > 0, \quad Q = Q^T \geq 0 \tag{24}$$

$$\bullet \; \text{rank} Q = 1 \tag{25}$$

$$\bullet \; \Psi_{\text{o}}^T \left[\begin{array}{c|c} E & 0 \\ \hline 0 & E \end{array}\right]^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \left[\begin{array}{c|c} E & 0 \\ \hline 0 & E \end{array}\right] \Psi_{\text{o}} < Q. \tag{26}$$

Remark 3. The data-based stabilization problem in Definition 1 is formulated in the form of a linear matrix inequality(LMI) with a rank constraint. However, the rank constraint in (25) still makes the entire data-based stabilization problem non-convex and NP-hard.

5.2. Feedback gains

In this subsection, we briefly examine how to determine the feedback gain from the solution of the data-based stabilization problem in Definition 1.

First, let us suppose that the matrix Q is obtained, then, from (22) and (23), the coefficient matrix Z can be readily determined as a basis of the nullspace of Q as

$$Z \in \ker W = \ker Q. \tag{27}$$

Next, by using Z obtained from (27), the feedback gain $K \in \mathbb{R}^{n+m+1}$ is similarly determined as a basis of the nullspace of the data matrix $\Psi_c^T(:, 2:n{+}m{+}2)$ as

$$K \in \ker \Psi_c^T(:, 2:n{+}m{+}2) = \ker(Z^T \Psi_{\text{o}}^T(:, 2:n{+}m{+}2)), \tag{28}$$

where $\Psi_c^T(:, 2:n{+}m{+}2)$ denotes the submatrix of Ψ_c^T whose columns consist of the 2nd to $(n+m+2)$-th column vectors of Ψ_c^T and similarly for $\Psi_{\text{o}}^T(:, 2:n{+}m{+}2)$. Note that the feedback gain K obtained from (28) is identical to the parameter vector $\theta_c^T(2:n{+}m{+}2)$ which specifies the feedback data space $\mathcal{D}_f$.

6. Computational Algorithm

In this section, we present a computational algorithm to solve the data-based stabilization problem in Definition 1 by using the LMI relaxation and linearization method [6], [7].

First, let us consider a factorization of the matrix Q as follows:

$$\begin{aligned} Q &= \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & q_{22} \end{bmatrix} \\ &= \begin{bmatrix} I_{n+m} & Q_{12}q_{22}^{-1} \\ 0 & I_1 \end{bmatrix} \begin{bmatrix} Q_{11} - Q_{12}(q_{22}^{-1})Q_{12}^T & 0 \\ 0 & q_{22} \end{bmatrix} \begin{bmatrix} I_{n+m} & 0 \\ q_{22}^{-1}Q_{12}^T & I_1 \end{bmatrix} \end{aligned} \tag{29}$$

where $q_{22} \in \mathbb{R}$ is a positive scalar. To satisfy the rank constraint in (25), the following condition has to be achieved:

$$Q_{11} - Q_{12}(q_{22}^{-1})Q_{12}^T = 0 \tag{30}$$

Hence, to achieve (30), we consider the minimization of the trace of $(q_{22}Q_{11} - Q_{12}Q_{12}^T)$ as

$$\min_{Q_{11}, Q_{12}, q_{22}} \text{Tr}\,(q_{22}Q_{11} - Q_{12}Q_{12}^T). \tag{31}$$

However, since the objective function to minimize in (31) includes nonlinear terms, we employ a linear approximation of $(q_{22}Q_{11} - Q_{12}Q_{12}^T)$ for given $\hat{Q}_{11}, \hat{Q}_{12}, \hat{q}_{22}$ as follows:

$$\begin{aligned} \min_{Q_{11}, Q_{12}, q_{22}} \text{Tr}\,\{ & \hat{q}_{22}Q_{11} + q_{22}\hat{Q}_{11} + \hat{q}_{22}\hat{Q}_{11} \\ & - (\hat{Q}_{12}Q_{12}^T + Q_{12}\hat{Q}_{12}^T + \hat{Q}_{12}\hat{Q}_{12}^T)\}. \end{aligned} \tag{32}$$

First, we start the computations by solving the LMI relaxed version of the data-based stabilization problem in Definition 1 from which the rank constraint in (25) is removed. Then, by using the solution Q of the LMI relaxed problem as an initial point, we iteratively solve the minimization problem in (32) with satisfying (24) and (26) until (25) is achieved. Concurrently, we also replace $\hat{Q}_{11}, \hat{Q}_{12}, \hat{q}_{22}$ with the new solution Q at each step. Here, a rough outline of the above algorithm is presented as follows:

Step 1: Find P, Q s.t. (24) and (26).
Set $Q_{11} = \hat{Q}_{11}, Q_{12} = \hat{Q}_{12}, q_{22} = \hat{q}_{22}$.

Step 2: Solve (32) s.t. (24) and (26).
Set $Q_{11} = \hat{Q}_{11}, Q_{12} = \hat{Q}_{12}, q_{22} = \hat{q}_{22}$.

Step 3: Check the stopping criterion; if it is satisfied, stop; otherwise go back to Step 2.

Remark 4. Note that the computational algorithm presented in this section does not guarantee the convergence to the global solution. In addition, there seems to be a strong dependence on the initial conditions in this algorithm.

7. Numerical Examples

In this section, we provide two numerical examples to demonstrate that we can design a stabilizing controller by using the proposed data-based stabilization condition as seen in Definition 1.

First, let us consider an example system (Σ_1) with $n = 3$ and $m = 1$ and with initial conditions given as follows:

$$\begin{aligned}\Sigma_1 :\ & y(k+3)+0.8y(k+2)+0.86y(k+1)-1.02y(k)\\ & = 1.5u(k+1)-1.2u(k), \quad k \in \mathbb{N}\\ & y(1)=1,\ y(2)=-1,\ y(3)=2,\ u(1)=1,\end{aligned}$$

whose dynamics is unstable as seen from its eigenvalues $\lambda(\Sigma_1)$ as

$$\lambda(\Sigma_1) = \{\ -0.7 \pm 1.1i \quad 0.6\ \}$$

In simulations, from the time instant $k = 1$ to $k = 20$, we first generate the open-loop input and output data by exciting the system Σ_1 with a random input sequence that has the values in the range between $[-1, 1]$. Then, from the sets of the open-loop input and output data, we can obtain the data matrix $\Psi_o(k)$ such that $\mathrm{rank}\Psi_o(k) = n+m+1 = 5$. For this basis matrix $\Psi_o = \Psi_o(k)$, we solve the data-based stabilization problem in Definition 1 by using the computational algorithm presented in Section 6.From the solution Q, as seen in (27) and (28), the feedback gain K_1 is obtained as follows:

$$K_1 = \begin{pmatrix} -0.50482913308313\\ 0.29542311898217\\ 0.20543708125289\\ 1.00000000000000\\ -0.12503461095847 \end{pmatrix},$$

which is normalized with respect to the fourth element. As a consequence, the control input sequence $u(k+m)$, $k \in \mathbb{N}$ can be calculated by using the feedback gain K_1 as seen in (6) and (8). Therefore, from the time instant $k = 21$, by applying this control input sequence to the system Σ_1, we obtain the closed-loop input and output data, as depicted in Fig. 1.

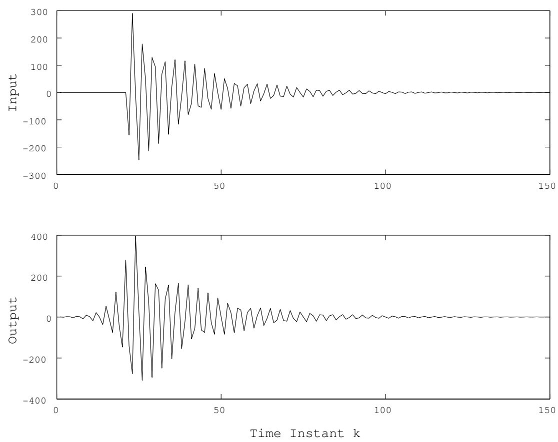

Fig. 1. Input and output response of Σ_1

Next, we consider an another example system (Σ_2) with $n = 4$ and $m = 2$ and with initial conditions given as follows:

$$\begin{aligned}\Sigma_2 :\ & y(k+4)+1.3y(k+3)-0.7y(k+2)-1.3y(k+1)+0.4y(k)\\ & = u(k+2)+0.5u(k+1)+1.2u(k), \quad k \in \mathbb{N}\\ & y(1)=1,\ y(2)=-1,\ y(3)=2,\ y(4)=-2,\ u(1)=1,\ u(2)=1.5.\end{aligned}$$

Also, the system Σ_2 has unstable dynamics as seen from its eigenvalues $\lambda(\Sigma_2)$ as

$$\lambda(\Sigma_2) = \{\ -1.1990 \pm 0.5102i \quad 0.8059 \quad 0.2922\ \}$$

Similarly, the feedback gain K_2 is also obtained as

$$K_2 = \begin{pmatrix} -0.43602624518044\\ 0.56363841077372\\ 0.86490839871053\\ -0.31330947311240\\ 1.00000000000000\\ 0.26982886413856\\ 0.93312798440503 \end{pmatrix},$$

and the closed-loop input and output data is depicted in Fig. 2.

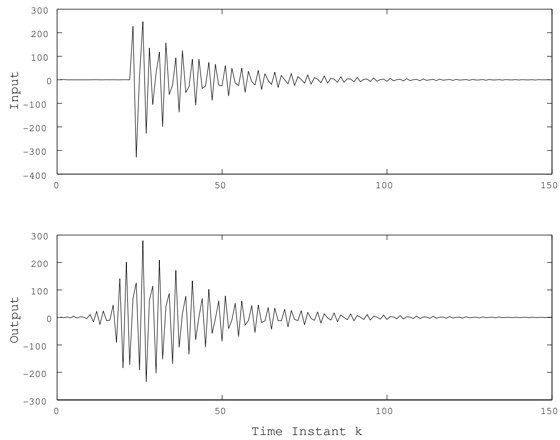

Fig. 2. Input and output response of Σ_2

These two simulations results show that all stabilizing controllers are obtained by using the data-based stabilization condition in Definition 1.

8. Conclusions

In this paper, we proposed the data-based control approach (or data space approach) as a new framework for feedback control design. In this approach, the notions of the open-loop and closed-loop data spaces are introduced to serve as the system representations in the data space, instead of employing the mathematical models. In addition, the relationship between these two data spaces that the closed-loop data space is contained in the open-loop data space as one of its subspace was established. In the end, by applying this relationship into the data-based stability condition for closed-loop systems, we derived the data-based stabilization condition for a linear time-invariant discrete-time system, which leads to a linear matrix inequality with a rank constraint.

References

[1] M. Ikeda and N. Hayashi, "Digital control based on input-output data," *Proceedings of the 16th SICE Symposium on Control Theory*, pp. 47–50, 1987. (in Japanese)

[2] M. Ikeda, Y. Fujisaki, and N. Hayashi, "A model-less algorithm for tracking control based on input-output data," *Nonlinear Analysis*, Vol. 47, No. 3, pp. 1953–1960, 2001.

[3] U. S. Park and M. Ikeda, "Data-based stability analysis for linear discrete-time systems," unpublished.

[4] L. Dai, *Singular Control Systems*, Lecture Notes in Control and Information Sciences, Vol. 118, Springer-Verlag, NY, 1989.

[5] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, London, 1998.

[6] L. El Ghaoui, F. Oustry, and M. AitRami, "A cone complementarity linearization algorithm for static output-feedback and related problems," *IEEE Transactions on Automatic Control*, Vol. 42, No. 8, pp. 1171–1176, 1997.

[7] T. Kiyama and E. Nishio, "Finite frequency property-based robust control analysis and synthesis," *Proceedings of the 2004 American Control Conference*, in press, 2004.

Appendix

A. Proof of Theorem 2

1)⇒2): First, since the system (E, A_c) in (14) is regular and causal, it follows that there exist nonsingular matrices $S, T \in \mathbb{R}^{(n+m+1)\times(n+m+1)}$ such that

$$SET = \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix}, \quad SA_cT = \begin{bmatrix} A_1 & 0 \\ 0 & I_1 \end{bmatrix}, \tag{33}$$

where $I_{n+m}, A_1 \in \mathbb{R}^{(n+m)\times(n+m)}$ and $I_1 \in \mathbb{R}$.

Suppose that the system (14) has an asymptotically stable equilibrium point, then there exists a positive definite symmetric matrix $P_1 \in \mathbb{R}^{(n+m)\times(n+m)}$ such that

$$A_1^T P_1 A_1 - P_1 < 0. \tag{34}$$

For some P_1 that satisfies (34), there exists a positive scalar $p_2 \in \mathbb{R}$ such that

$$x_1^T(A_1^T P_1 A_1 - P_1)x_1 + x_2^T p_2 x_2 < 0, \quad \forall x_1 \neq 0, \ x_2 = 0. \tag{35}$$

Then, (35) can be rewritten as

$$\begin{aligned}
&\begin{pmatrix} x_1^T & x_2^T \end{pmatrix} \begin{bmatrix} A_1^T & 0 \\ 0 & I_1 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & I_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \\
&\begin{pmatrix} x_1^T & x_2^T \end{pmatrix} \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} < 0 \\
&\forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0 \ \text{ s.t. } \ \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0 \ \text{ and } \ x_2 = 0.
\end{aligned} \tag{36}$$

Also, we have the following equivalent condition to (36):

$$\begin{aligned}
&\begin{pmatrix} x_1^T(k+1) & x_2^T(k+1) \end{pmatrix} \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} \\
&- \begin{pmatrix} x_1^T(k) & x_2^T(k) \end{pmatrix} \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} < 0 \\
&\forall \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \neq 0, \ k \in \mathbb{N} \ \text{ s.t. } \ \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \neq 0 \\
&\quad \text{and } \begin{bmatrix} I_{n+m} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_1 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}.
\end{aligned} \tag{37}$$

From this, if we define a positive definite symmetric matrix $P \in \mathbb{R}^{(n+m+1)\times(n+m+1)}$ and a vector $x_c \in \mathbb{R}^{(n+m+1)}$ as

$$P = S^T \begin{bmatrix} P_1 & 0 \\ 0 & p_2 \end{bmatrix} S, \quad x_c(k) = T \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}, \tag{38}$$

then, from (33) and (37), we can readily obtain the condition 2).

2)⇒1): For a positive definite symmetric matrix P, let us consider a Lyapunov function V as

$$V(x_c) = x_c^T(E^T PE)x_c > 0 \quad \forall x_c \neq 0 \ \text{ s.t. } Ex_c \neq 0. \tag{39}$$

As seen in the condition 2), since the difference of V with respect to k along any trajectory of the system (14) is negative definite ($\nabla V < 0$), the system has an asymptotically stable equilibrium point.

2)⇔3): First, the condition 2) can be rewritten as follows:

$$\begin{aligned}
&\exists P = P^T > 0 \ \text{ s.t.} \\
&\begin{pmatrix} x_c^T(k+1) & x_c^T(k) \end{pmatrix} \begin{bmatrix} E^T PE & 0 \\ 0 & -E^T PE \end{bmatrix} \begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix} < 0 \\
&\forall \begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix} \neq 0, \ k \in \mathbb{N} \ \text{ s.t.} \\
&\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix} \neq 0 \ \text{ and } \ \begin{bmatrix} E & -A_c \end{bmatrix} \begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix} = 0.
\end{aligned} \tag{40}$$

In (40), the constraints on the vector $\begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix}$ specify a certain data space that is described as

$$\Omega = \left\{ \begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix} \in \mathbb{R}^{2(n+m+1)} \;\middle|\; \begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix} \in \begin{bmatrix} E & -A_c \end{bmatrix}^{\perp} \setminus \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}^{\perp} \right\},$$

where $\dim \Omega = \text{rank} E = n + m$.

From (17), it is easily seen that the data matrices $\Phi_c(k+1)$ and $\Phi_c(k)$ construct a subspace of $\begin{bmatrix} E & -A_c \end{bmatrix}^{\perp}$. Also, it is obvious from (1) and (6) that there are no such input and output data sets that lie in the nullspace of E, which means that the input and output data describe only the dynamic behavior of the system (14). Hence, we have $E\Phi_c(k+1) \neq 0$ and $E\Phi_c(k) \neq 0$. Now, for some k_0, if we obtain the data matrix $\Phi_c(k_0)$ of full column rank that satisfies the rank condition as

$$\text{rank} \begin{bmatrix} \Phi_c(k_0+1) \\ \Phi_c(k_0) \end{bmatrix} = n + m, \tag{41}$$

then, the data matrices in (41) becomes a basis matrix of the subspace Ω. From this, it follows that all the vectors $\begin{pmatrix} x_c(k+1) \\ x_c(k) \end{pmatrix}$ in (40) can be expressed as a linear combination of the column vectors of the basis matrix in (41), hence we have the following equivalent condition to (40):

$$\begin{aligned}
&\exists P = P^T > 0 \ \exists k_0 \ \text{ s.t.} \\
&\begin{bmatrix} \Phi_c^T(k_0+1) & \Phi_c^T(k_0) \end{bmatrix} \begin{bmatrix} E^T PE & 0 \\ 0 & -E^T PE \end{bmatrix} \begin{bmatrix} \Phi_c(k_0+1) \\ \Phi_c(k_0) \end{bmatrix} < 0,
\end{aligned} \tag{42}$$

for the data matrix $\Phi_c(k_0)$ of rank $(n+m)$.

Furthermore, the following relation holds for the data matrix $\Psi_c(k_0)$ of rank $(n+m)$:

$$\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} \Phi_c(k_0+1) \\ \Phi_c(k_0) \end{bmatrix} = \left[\begin{array}{c:c} E & 0 \\ \hdashline 0 & E \end{array}\right] \Psi_c(k_0), \tag{43}$$

where

$$\left[\begin{array}{c:c} E & 0 \\ \hdashline 0 & E \end{array}\right] \in \mathbb{R}^{2(n+m+1)\times(n+m+2)}. \tag{44}$$

Thus, by substituting (43) into (42), we can finally obtain the condition 3). The converse is also true, hence this completes 2)⇔3).