Flexure Analysis of Inertial Navigation Systems

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Abstract: Ring Laser Gyroscopes used as navigational sensors inherently experience a lock-in region, where very low rotational rates are not measurable. Most RLG manufacturers use a mechanical dither motor that applies a small oscillatory rotational motion larger than this region to resolve this problem. Any input acceleration that bends this dithering axis causes flexure error, which is a noncommutative error that can not be compensated by simply using integrated gyro sensor output. This paper introduces noncommutative error equations that define attitude errors caused by flexure errors. In this paper, flexure error is classified as sensor level error if the sensing axis coincides with the dithering axis and as system level error if the two axes do not coincide. The relationship between gyro output and the rotation vector is introduced and is used to define the coordinate transformation matrix and angular motion. Equations are derived for both sensor level and system level flexure error analysis. These equations show that RLG based INS attitude error caused by flexure is directly proportional to time, amount of input acceleration and the dynamic frequency of the vehicle.

Keywords: Ring Laser Gyro, Coning Motion, Flexure Error, Non-commutative Error, Attitude Error.

1. INTRODUCTION

The RLG used in strapdown configuration has become the backbone of most current commercial and military navigation systems. Strapdown RLGs have a wide dynamic range and bandwidth, high accuracy and reliability, and relatively low acquisition costs which makes them extremely attractive for navigation systems[1]. Moreover, RLGs can provide sufficiently accurate angular rate and acceleration information to be used in flight control[2].

The lock-in condition of RLG occurs when the two laser beams in the cavity cease to oscillate at different frequencies and assume the same frequency at very low input rotation rates[3]. There is no change in the interference pattern for input rates in this lock-in region, resulting in no output signal. One of the most common methods used to alleviate the lock-in problem is the use of mechanical oscillation[4]. Mechanical dithering consists of applying angular vibrations to the entire cavity at high frequency but at low amplitude and through small angles, thereby avoiding low frequency outputs. However, coupling of dither motions between the three axes of the instrument block leads to coning motion of the entire assembly and bending of dithering axis caused by any input acceleration introduces flexure error. These coning and flexure errors lead to apparent gyro bias errors in the navigation system[5].

Coning errors can be compensated by removing gyro block dither motion, which can be accomplished by techniques such as dither stripping, dither trapping and digital filtering. In addition, arithmetic calculations of the errors by any attitude algorithm can compensate coning errors[6]. However for flexure errors, efforts have been more focused on structural design techniques that minimize dither axis bending.

This paper introduces non-commutative error equations that define attitude errors caused by flexure errors. The relationship between gyro output and the rotation vector is introduced, and is used to define the coordinate transformation matrix and angular motion. In this paper, flexure error is classified as sensor level error if the sensing axis coincides with the dithering axis and as system level error if the two axes do not coincide. Equations are derived for both sensor level and system level flexure error analysis.

2. NONCOMMUTATIVITY ERROR

When the navigation computer calculates the vehicle attitude, computational errors are caused by both commutative and noncommutative errors. The commutative error that is derived from integrating gyro outputs can be reduced by increasing the frequency of computation updates. However, since the noncommutative error is caused by non-inertially measurable angular motion, the compensation is never perfect, regardless of attitude update rates. This causes noncommutativity to be one of the major error sources in solving the attitude calculation equation. The theoretical basis for the noncommutative error is established by Bortz[7].

To describe the generalized solution of the noncommutative error, the rotation frame is defined as shown in Fig. 1. In Fig. 1, r^B is an arbitrary vector fixed in the body frame and r^R is the reference frame. Suppose that r^B and r^R is coincided at $t = t_0$. Then,

$$r_0^R = r_0^B \tag{1}$$

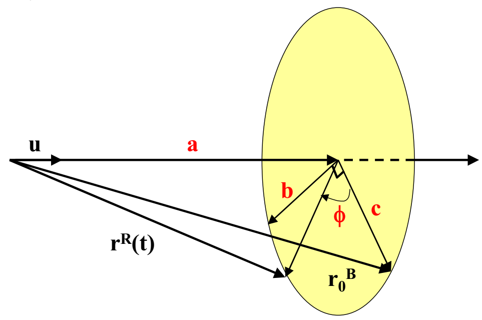

Fig. 1 The Frame Rotation

If the body frame rotates with respect to the reference frame through the angle ϕ in Fig. 1, the relationship between the two frames is

$$r^R(t) = C_B^R r^B(t) \tag{2}$$

where, $r^B(t) = r_0^B$, that is, $r^B(t)$ is always fixed in the body frame. So Equation (2) can be given by

$$r^R(t) = C_B^R r_0^B \tag{3}$$

Using the vector relationships used in constructing figure 1, (3) can be solved for $r^R(t)$ as follows

$$r^R(t) = \mathbf{a} + \mathbf{b}\sin\phi + \mathbf{c}\cos\phi \tag{4}$$

where, **a**, **b** and **c** are defined as follows

$$\mathbf{a} = \mathrm{u\,u^T\,r_0^B},\quad \mathbf{b} = [\mathrm{u}\times]\mathrm{r_0^B},\quad \mathbf{c} = \mathrm{r_0^B} - \mathrm{u\,u^T\,r_0^B} \tag{5}$$

By substituting the variables of (5) in (4), the following equation may be easily derived.

$$r^R(t) = \{\mathrm{uu^T}(1-\cos\phi) + \mathrm{I}\cos\phi + [\mathrm{u}\times]\sin\phi\} r_0^B \tag{6}$$

The unit vector u defining the rotation axis in figure 1 is given by

$$u = \frac{\bar{\phi}}{\phi}$$

Then comparing (6) with (3) leads to the transformation matrix

$$C_B^R = I + \frac{\sin\phi}{\phi}[\bar{\phi}\times] + \frac{1-\cos\phi}{\phi^2}[\bar{\phi}\times]^2 \tag{7}$$

Equation (7) shows that the transformation matrix can be calculated by the rotation vector $\bar{\phi}$.

From the derivative of the transformation matrix, $\dot{C}_B^R = C_B^R[\omega\times]$, the rotation vector differential equation is given by

$$\dot{\bar{\phi}} = \omega + \frac{1}{2}\bar{\phi}\times\omega + \frac{1}{\phi^2}\left(1 - \frac{\phi\sin\phi}{2(1-\cos\phi)}\right)\bar{\phi}\times(\bar{\phi}\times\omega) \tag{8}$$

Utilizing (6), the angular rate vector ω measured by gyro is explicitly defined by

$$\omega = \dot{\bar{\phi}} - \frac{1-\cos\phi}{\phi^2}\bar{\phi}\times\dot{\bar{\phi}} + \frac{1}{\phi^2}\left(1 - \frac{\sin\phi}{\phi}\right)\bar{\phi}\times(\bar{\phi}\times\dot{\bar{\phi}}) \tag{9}$$

The relationships defined by (7) and (9) show that if the rotation vector representing sensor or system dynamics is provided, we can determine the angular rate of gyros and the transformation matrix between the reference frame and the body frame. Generally, the rotation vector can represent the dynamics between the reference frame and the body frame more simply than the angular rate vector that accounts for correlations of the three axes.

In this paper, we define the rotation vector for representing the flexure and analyze the characteristic of the noncommutative error by utilizing equations (7) and (9).

3. SENSOR LEVEL FLEXURE ERROR

If an acceleration is applied to the RLG with dithering motion, the dither axis of the RLG will bend proportional to the amount of the input acceleration. This type of error is defined as the sensor level flexure error. The sensor level means that an axis bending happens to the dithering axis without changing an input axis of the vehicle rate.

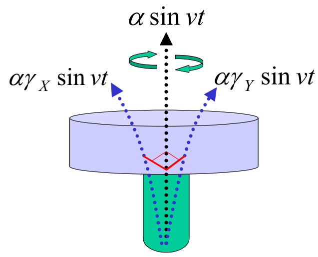

Fig. 2 Bending of the dither axis under an acceleration

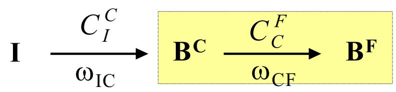

Fig. 3 Varying of frames caused by flexure

Figure 2 depicts the sensor level flexure and Figure 3 represents the varying of frames. In figure 2, α is the amplitude of the sinusoidal dithering motion, ν is the dithering rate and γ is the flexure angle given by

$$\gamma = \beta_f \cdot a_{input} \tag{10}$$

where β_f is the flexure constant representing the stiffness of the dithering axis and a_{input} is the input acceleration.

In figure 3, frames are defined as

- I : Inertial frame
- B^C : changed frame caused by the coning motion of ISA
- B^F : changed frame caused by the sensor level flexure

From the relationship depicted in figure 3, the gyro output is found to be

$$\omega_{IF} = \omega_{CF} + C_C^F\omega_{IC} \tag{11}$$

In (11), we firstly derive the angular rate ω_{CF} with the aid of the rotation vector that represents the dithering motion as

$$\bar{\phi} = \begin{bmatrix} \alpha\gamma_X\sin\nu t \\ \alpha\gamma_Y\sin\nu t \\ \alpha\sin\nu t \end{bmatrix} \tag{12}$$

Substituting (12) and its derivative into (9), consolidating terms and reducing, leads to the angular rate vector

$$\omega_{CF} = \begin{bmatrix} \alpha\nu\gamma_X\cos\nu t \\ \alpha\nu\gamma_Y\cos\nu t \\ \alpha\nu\cos\nu t \end{bmatrix} \tag{13}$$

Secondly, the transformation matrix C_C^F is obtained by utilizing (7). For substituting (12) into (7), the product of the rotation vector (12) and its square are given by

$$[\bar{\phi}\times] = \begin{bmatrix} 0 & -\alpha\sin\nu t & \alpha\gamma_Y\sin\nu t \\ \alpha\sin\nu t & 0 & -\alpha\gamma_X\sin\nu t \\ -\alpha\gamma_Y\sin\nu t & \alpha\gamma_X\sin\nu t & 0 \end{bmatrix} \tag{14}$$

$$[\bar{\phi}\times]^2 = \begin{bmatrix} -\alpha^2(1+\gamma_Y^2)\sin^2\nu t & \alpha^2\gamma_X\gamma_Y\sin^2\nu t & \alpha^2\gamma_X\sin^2\nu t \\ \alpha^2\gamma_X\gamma_Y\sin^2\nu t & -\alpha^2(1+\gamma_Y^2)\sin^2\nu t & \alpha^2\gamma_Y\sin^2\nu t \\ \alpha^2\gamma_X\sin^2\nu t & \alpha^2\gamma_Y\sin^2\nu t & -\alpha^2(\gamma_X^2+\gamma_Y^2)\sin^2\nu t \end{bmatrix} \tag{15}$$

The magnitude of the rotation vector is found to be

$$\phi = \alpha\sin\nu t\sqrt{1+\gamma_X^2+\gamma_Y^2} \tag{16}$$

Substituting (14), (15) and (16) into (7), and neglecting α 's higher order terms with the aid of small angle approximation, leads to the transformation matrix

$$C_C^F \approx I + \alpha \sin vt \begin{bmatrix} 0 & -1 & \gamma_Y \\ 1 & 0 & -\gamma_X \\ -\gamma_Y & \gamma_X & 0 \end{bmatrix} \tag{17}$$

Thirdly, ω_{IC} is provided by the dynamic relationship between the inertial frame and the IMU frame. For sensor level analysis, let's consider an IMU dynamics under sinusoidal dithering motion without vehicle movement. In this case, the IMU will sweep out a cone from the combined effects of the dithering motion of the three gyros. The angular rate under the coning motion is given by

$$\omega_{IC} = \begin{bmatrix} v\varepsilon \cos(vt+\tau_0) \\ -v\varepsilon \sin(vt+\tau_0) \\ \dfrac{v\varepsilon^2}{2} \end{bmatrix} \tag{18}$$

where, ε is the amplitude of coning motion and τ_0 is the phase difference between the z-axis and tangential axes.

Substituting (13), (17) and (18) into (11) and integrating the result, leads to the incremental angle output

$$\Delta\theta_C = \delta\theta_C^{\text{Constant}} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{v\varepsilon^2}{2} t_n \end{bmatrix} + \begin{bmatrix} \dfrac{\alpha v \varepsilon}{2} \cos \tau_0 t_n \\ -\dfrac{\alpha v \varepsilon}{2} \sin \tau_0 t_n \\ 0 \end{bmatrix} \tag{19}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \dfrac{\alpha v \varepsilon}{2} (\gamma_Y \sin \tau_0 - \gamma_X \cos \tau_0) t_n \end{bmatrix} \tag{20}$$

where, $\delta\theta_C^{\text{Constant}}$ is the bounded terms regardless of time.

From the gyro output given by (19) and (20), the system error is increased with time when sensor level flexure and the IMU coning motion simultaneously occurs. The second and third terms on the right-hand side of (19) are considered as the coning drift error and equation (20) is considered as the sensor level flexure error.

If the dithering frequency is 400Hz, the dithering amplitude α is 0.01[deg], the coning amplitude ε is 0.01[deg], the flexure angle γ is 100[arcsec] and τ_0 is 0[deg], then the flexure error calculated by (20) is

$$\Delta\theta_C^{Flexure} \approx 0.002 \text{ [deg/} hr\text{]} \tag{21}$$

The above value is small enough to avoid any specific compensation. Therefore, the sensor level flexure error does not appear to be a great error source. However, γ is varies with input acceleration. If the input acceleration is 10[g], then the flexure constant must be 2[arcsec/g] to result in the same flexure error as in (21). So the RLG designer must consider the stiffness of the dither axis to reduce this flexure error to an acceptable level.

4. SYSTEM LEVEL FLEXURE ERROR

When an angular rate caused by the motion of the vehicle is provided in the input axis of the RLG, the RLG sensing axis coincides with the input axis of the motion.

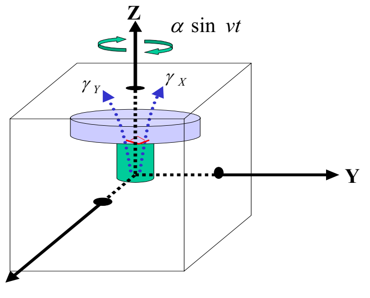

Fig. 4 The input axis of the vehicle motion and the sensing axis of RLG

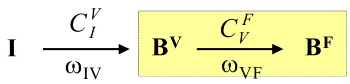

Fig. 5 Varying of frames caused by the flexure

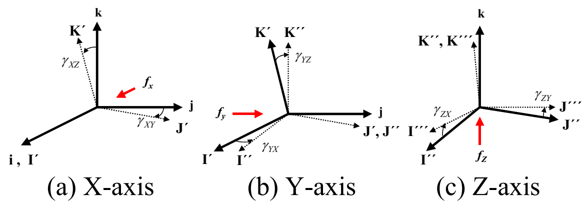

Fig. 6 Frame variations by the acceleration

Initially as shown in figure 4, the X, Y, Z axes are aligned to the input axes of angular rate vectors, and also coincide with the body frame. However, when an input acceleration is present, a bend represented by γ occurs in the sensing axis, while the input axis remains constant. If an acceleration is applied to the X or Y axis, the Z-axis gyro measures an angular rate different from the actual input angular motion due to this flexure in the sensing axis. The relationship between the body frames of a system experiencing flexure is shown in figure 5. In this figure, the frames are defined as

- I : Inertial frame
- B^V : Body frame is aligned with the vehicle
- B^F : Changed frame caused by the flexure

To obtain the gyro output using the relationship depicted in figure 5, we consider unit vectors that are aligned with the sensing axes in figure 6. Figure 6(a) shows the Y and Z deformation caused by X axis acceleration.

From figure 6, relationship between initial unit vectors and unit vectors changed by input acceleration can be expressed as follows

$$\begin{bmatrix} I' \\ J' \\ K' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \sin\gamma_{XY} & \cos\gamma_{XY} & 0 \\ \sin\gamma_{XZ} & 0 & \cos\gamma_{XZ} \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} I'' \\ J'' \\ K'' \end{bmatrix} = \begin{bmatrix} \cos\gamma_{YX} & \sin\gamma_{YX} & 0 \\ 0 & 1 & 0 \\ 0 & \sin\gamma_{YZ} & \cos\gamma_{YZ} \end{bmatrix} \begin{bmatrix} I' \\ J' \\ K' \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} I''' \\ J''' \\ K''' \end{bmatrix} = \begin{bmatrix} \cos\gamma_{ZX} & 0 & \sin\gamma_{ZX} \\ 0 & \cos\gamma_{ZY} & \sin\gamma_{ZY} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I'' \\ J'' \\ K'' \end{bmatrix} \tag{24}$$

Flexure angles are very small in the above equations, so equations (22), (23) and (24) with the aid of small angle approximation lead to

$$\begin{bmatrix} I''' \\ J''' \\ K''' \end{bmatrix} = \begin{bmatrix} 1 & \gamma_{YX} & \gamma_{ZX} \\ \gamma_{XY} & 1 & \gamma_{ZY} \\ \gamma_{XZ} & \gamma_{YZ} & 1 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \quad (25)$$

The transformation matrix representing the frame change caused by flexure is obtained as follows

$$C_V^F = I + \begin{bmatrix} 0 & \gamma_{YX} & \gamma_{ZX} \\ \gamma_{XY} & 0 & \gamma_{ZY} \\ \gamma_{XZ} & \gamma_{YZ} & 0 \end{bmatrix} \quad (26)$$

In figure (5), the vector ω_{IV} is the measurement of the angular rate between the inertial frame and the vehicle body frame. Because the vehicle is undergoing many motions in flight, the definition of the ω_{IV} can vary. In this paper, we assume that the vehicle is under coning motion, so the angular rate vector becomes (27) by definition of coning motion.

$$\omega_{IV} = C \begin{bmatrix} v\varepsilon_X \cos vt \\ -v\varepsilon_Y \sin vt \\ v\varepsilon_X\varepsilon_Y / 2 \end{bmatrix} \quad (27)$$

Using (26) and (27), the angular rate vector that is depicted in figure 5 is obtained as

$$\omega_{IF} = \begin{bmatrix} v\varepsilon_X \cos vt \\ -v\varepsilon_Y \sin vt \\ v\varepsilon_X\varepsilon_Y / 2 \end{bmatrix} + \frac{v\varepsilon_X\varepsilon_Y}{2} \begin{bmatrix} \gamma_{ZX} \\ \gamma_{ZY} \\ 0 \end{bmatrix} + \begin{bmatrix} -\gamma_{YX} v\varepsilon_Y \sin vt \\ \gamma_{XY} v\varepsilon_X \cos vt \\ \gamma_{XZ} v\varepsilon_X \cos vt - \gamma_{YZ} v\varepsilon_Y \sin vt \end{bmatrix} \quad (28)$$

The first term on the right-hand side of (28) is the rate vector of the coning motion, the second term is the rate vector that projects the coning drift rate on the X and Y axes, and third term is the rate vector projecting the sinusoidal motion into 3-axes.

To analyze the system level flexure error provided by (28), the flexure angle γ_{ij} must be defined and the study focused on two points.

1) if an input acceleration is constant.

2) if an input acceleration is sinusoidal.

Firstly, if an input acceleration is constant, the increased error with time is the second term on the right-hand side of (28). Let's consider the case where coning drift is 5[deg/hr] and the γ_{ij} is 50[arcsec](=2[arcsec/g] × 10[g]). Then the attitude drift of the X axis is 0.0012[deg/hr].

Secondly, if an input acceleration is sinusoidal, the input acceleration is defined as follows

$$\begin{aligned} f_X &= g_X \sin(vt + \tau_X) \\ f_Y &= g_Y \sin(vt + \tau_Y) \\ f_Z &= g_Z \sin(vt + \tau_Z) \end{aligned} \quad (29)$$

By substituting (29) into (28) and using trigonometric functions, we can define the angular rate vector in case of sinusoidal acceleration

$$\omega_{IF}^{FS} = \delta\omega_{IF}^{FS-\mathrm{Constant}} + \begin{bmatrix} -\frac{1}{2}\beta_{YX} g_Y v\varepsilon_Y \cos\tau_Y \\ \frac{1}{2}\beta_{XY} g_X v\varepsilon_X \sin\tau_X \\ \frac{1}{2}(\beta_{XZ} g_X v\varepsilon_X \sin\tau_X - \beta_{YZ} g_Y v\varepsilon_Y \cos\tau_Y) \end{bmatrix} \quad (30)$$

where, $\delta\omega_{IF}^{FS-\mathrm{Constant}}$ is the bounded terms regardless of time.

Let's consider a 2g-60Hz external excitation. Using the result in (30) and assuming β_{ij} = 0.3[arcsec/g], ε = 100 [arcsec] and τ = 0, the attitude drift of the X axis caused by the flexure error is 0.055[deg/hr]. This value is obviously unacceptable for navigation performance. This result show that the gyro performance under severe vibrations is considerably decreased.

5. CONCLUSION

In this paper, we have developed flexure error equations of RLG based strapdown navigation systems for both as sensor level and system level flexure error. We have shown that the sensor level error and the system level error with a constant input acceleration are not to be great, compared with the navigation performance, but the system level error with a sinusoidal input acceleration is to be severe error.

ACKNOWLEDGMENTS

This research is supported by Agency for Defense Development and KISTEP Space and Basic Science Programs (M10336000021-03D0500-02810).

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