The MMAP\M\c System with Heterogeneous Servers and Retrial

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Abstract

Multi-server Markovian retrial model with heterogeneous servers is analyzed. Arriving customers constitute the MMAP (Marked Markovian Arrival Process). Distribution of the primary customers among the servers is performed randomly depending on the type of a customer and the number of the server. Customers from the orbit have the exponential service time distribution with a parameter depending on the server only. The choice of the server for a retrial is made randomly as well. Multidimensional continuous time Markov chain describing operation of the model is investigated by means of reducing to asymptotically quasi-toeplitz Markov chains.

1. Introduction

Theory of multi-server retrial queues is developed mainly for the case of identical full available servers and the stationary Poisson arrival process. The model under consideration is more general in the threefollowing respects: the servers are allowed to be heterogeneous; the input flow is heterogeneous as well; distribution of primary and repeated customers among the servers is stochastic. The systems of such a kind can model, e.g., operation of some inquiryoffice having several different locations or phone numbers. So, their investigation is interesting from both theoretic and application points of view.

2. Description of the model

We consider the retrial queueing model having c servers. The arriving flow constitutes the MMAP(Marked Markovian Arrival Process). The MMAP is the natural extension of the well-known MAP to the case of heterogeneous customers. The behavior of the MMAP is governed by underlying continuous time Markov chain $\eta_t, t \geq 0, \eta_t = \overline{0,W}$. The intensities of transitions of this chain form the matrices $D_k, k = \overline{0,K}$. Transitions, which are governed by the matrix D_0, do not lead to the customers arrival while the transitions governed by the matrix D_k cause generation of a customer having the type k, $k = \overline{1,K}$. Denote by θ the row vector of the stationary probabilities of the chain $\eta_t, t \geq 0$. It satisfies equations $\theta \sum_{k=0}^{K} D_k = \vec{0}, \theta e = 1,$ where $\vec{0}$ is the row vector consisting of zeroes, e is the column vector consisting of ones. The arriving customer of type k is directed to the server number m with probability $q_m^{(k)}, \sum_{m=1}^{c} q_m^{(k)} = 1, m = \overline{1,c}, k = \overline{1,K}$. If the server is idle at the arrival epoch, the customer is processed during the exponentially distributed time with the rate $\mu_m^{(k)}, m = \overline{1,c},\ k = \overline{1,K}$ and leaves the system afterwards. But if the selected server is busy at the arrival epoch, the customer moves to so called orbit to try its luck later on (although some another servers can be idle at this arrival epoch).

Each customer from the orbit generates the attempts to get service independently of other customers and the inter-attempt intervals are exponentially distributed with a parameter $\alpha, \alpha \rangle 0$.

3. Description of the Markov chain

Consider the process $\zeta_t = \{i_t, v_t^{(1)}, \dots, v_t^{(1)}, \eta_t\}, t \ge 0$, where $i_t, i_t \ge 0$, is the number of customers on the orbit at epoch $t, v_t^{(m)}$ is the state of the r^{th} server at epoch $t, m = \overline{1,c}$ and defined as follow:

$$v_t^r = \begin{cases} 0, & \textit{if server is idle} \\ k, & \textit{if server processes the primal customer of type } k, k = \overline{1,K} \\ K+1, & \textit{if server processes the customer from the orbit at epoch } t \end{cases}$$

$\eta_t, \eta_t = \overline{0,W}$ is the state of the MMAP underlying process at epoch t. It is easy to see that the process $\zeta_t, t \ge 0$ in the multi-dimensional continuous time Markov chain. Denote the stationary distribution of this chain as

$$p(i, v_1, \dots, v_c, \eta) = \lim_{t \to \infty} \{i_t = i, v_t^{(1)} = v_1, \dots, v_t^{(c)} = v_c, \eta_t = \eta\},$$

$$i \ge 0, v_m = \overline{0, K+1}, m = \overline{1,c}, \eta = \overline{0,W}$$
(1)

Enumerate the states of the chain $\zeta_t, t \ge 0$ in the lexicographic order and form the row vectors $\vec{p}_i$ of probabilities corresponding to the state $i, i \ge 0$, of the component i_t. Dimension of these vectors is equal to $M = (K+2)^c W$ where $\overline{W} = W + 1$. Form also the macro-vector $\vec{p} = (\vec{p}_0, \vec{p}_1, \dots, \vec{p}_i, \dots)$.

Lemma 1. *The vector $\vec{p}$ is calculated as the unique solution to the system*

$$\vec{p}Q = \vec{0}, \vec{p}e = 1, \qquad (2)$$

where the infinitesimal generator Q of the Markov chain $\zeta_t, t \ge 0$ has the form:

$$Q = \begin{pmatrix} Q_{0,0} & Q_{0,1} & O_c & O_c & \cdots \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & O_c & \cdots \\ O_c & Q_{2,1} & Q_{2,2} & Q_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (3)$$

where the blocks $Q_{i,j}$ are calculated as:

$$Q_{i,j} = A_c - i\alpha B_c, i \ge 0, Q_{i,i-1} = i\alpha L_c, i \ge 1, Q_{i,i+1} = S_c, i \ge 0, \qquad (4)$$

where the matrices A_c, B_c, L_c, S_c of dimension M are calculated from the following recurrent relations:

$$A_0 = D_0, B_0 = I_0, L_0 = O_0, S_0 = O_0,$$

$$A_{m+1} = \begin{pmatrix} A_m & I \otimes D_1 q_{m+1}^{(1)} & I \otimes D_2 q_{m+1}^{(2)} & \dots & I \otimes D_K q_{m+1}^{(K)} & O_m \\ \mu_{m+1}^{(1)} I_m & A_m - \mu_{m+1}^{(1)} I_m & O_m & \dots & O_m & O_m \\ \mu_{m+1}^{(2)} I_m & O_m & A_m - \mu_{m+1}^{(2)} I_m & \dots & O_m & O_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{m+1}^{(K)} I_m & O_m & O_m & \dots & A_m - \mu_{m+1}^{(K)} I_m & O_m \\ \mu_{m+1} I_m & O_m & O_m & \dots & O_m & A_m - \mu_{m+1} I_m \end{pmatrix},$$

$$B_{m+1} = diag\{B_m, B_m - \varphi_{m+1} I_m, \dots, B_m - \varphi_{m+1} I_m\},$$

$$L_{m+1} = \begin{pmatrix} L_m & O_m & O_m & \dots & O_m & \varphi_{m+1} I_m \\ O_m & L_m & O_m & \dots & O_m & O_m \\ O_m & O_m & L_m & \dots & O_m & O_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O_m & O_m & O_m & \cdots & O_m & L_m \end{pmatrix}$$

$$S_{m+1} = diag\{S_m, S_m + I_m \otimes \sum_{k=1}^{K} D_k q_{m+1}^{(k)}, \dots, S_m + I_m \otimes \sum_{k=1}^{K} D_k q_{m+1}^{(k)}\}, m = \overline{0, c-1},$$

where I_m is the identity matrix of dimension $(K+2)^m$, I_m is the identity matrix of dimension $(K+2)^m \overline{W}$, O_m is zero matrix of dimension $(K+2)^m \overline{W}, m = \overline{0,c}$, $\otimes$ is the sign of Kroneker product of matrices.

4. Stability condition

Theorem 1. *The stationary state distribution of the queue exists if the following inequality holds true:*

$$\rho = \frac{\theta \sum_{k=1}^{K} D_k e}{\sum_{m=1}^{c} \mu_m} < 1. \tag{5}$$

Proof. Aiming to investigate the Markov chain $\zeta_t = \{i_t, v_t^{(1)}, \dots, v_t^{(c)}, \eta_t\}, t \geq 0$, we exploit the results by [1,2] derived for the multi-dimensional discrete time Markov chains. Consider the discrete time Markov chain $\zeta_n, n \geq 1$ embedded at the epochs of all transitions of the Markov chain $\zeta_t = \{i_t, v_t^{(1)}, \dots, v_t^{(c)}, \eta_t\}, t \geq 0$. It is easy to verify that the one-step transition probability matrix P of Markov chain $\zeta_n, n \geq 1$ has the form

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & O_c & O_c & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & O_c & \cdots \\ O_c & P_{2,1} & P_{2,2} & P_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where, $P_{i,i-1} = i\alpha (F_c + i\alpha \hat{I})^{-1} L_c, i \geq 1$,

$P_{i,i} = I + (F_c + i\alpha \hat{I})^{-1}(A_c - i\alpha B_c), i \geq 0,$

$$P_{i,i+1} = (F_c + i\alpha \hat{I})^{-1} S_c, i \geq 0, \tag{6}$$

the matrix $\hat{I}$ is obtained from the identity matrix I by means of replacing the diagonal entries corresponding to zero diagonal entries by the matrix B_c with 0, the matrix F_c of dimension M is calculated from the recursion:

$$F_{m+1} = diag\{F_m, F_m + \mu_{m+1}^{(1)} I_m, \dots, F_m + \mu_{m+1} I_m\}, m = \overline{0, c-1}$$

the matrix F_0 is the diagonal one and its diagonal entries coincide to the corresponding diagonal entries of the matrix D_0.

Denote $\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, \dots)$ the row vector of stationary probabilities of the Markov chain $\xi_n, n \geq 1$. It can be shown that if the stationary distribution of this chain exists than the stationary distribution of original continuous Markov chain exists as well and is calculated as

$$\vec{p}_i = \bar{c} \vec{\pi}_i R_i^{-1}, i \geq 0, \tag{7}$$

where $R_i = F_c + i\alpha \hat{I}$, the constant $\bar{c}$ is found from normalization condition. Analyzing the structure of the matrix P, we can make sure that the discrete time Markov chain $\xi_n, n \geq 1$ belongs to the class of asymptotically quasi-toeplitz Markov chains [2]. It is shown in [2] that the stationary state distribution existence condition coincides with one valid for the limiting quasi-toeplitz Markov chain. This chain is defined in our case by blocks $\tilde{P}_{i,l}, l = i-1, i, i+1$ which are the limits of blocks defined by formula (6) when i tends to infinity and have the form

$$\tilde{P}_{i,i-1} = \tilde{Y}_0 = \hat{I} L_c, i \geq 1,$$

$$\widetilde{P}_{i,i} = \widetilde{Y}_1 = I - \hat{I}B_c + \bar{I}F_c^{-1}A_c,$$

$$\widetilde{P}_{i,i+1} = \widetilde{Y}_2 = IF_c^{-1}S_c, i \geq 0.$$

Introduce the matrix generating function $\widetilde{Y}(z) = \widetilde{Y}_0 + \widetilde{Y}_1 z + \widetilde{Y}_2 z^2, |z| \leq 1$. As follows from [1], stability condition for the Markov chain $\xi_n, n \geq 1$ has the form:

$$\vec{X}\widetilde{Y}(1)e < 1, \quad (8)$$

where the vector $\vec{X}$ is the unique solution to the system

$$\vec{X}\widetilde{Y}(1) = \vec{X}, \vec{X}e = 1. \quad (9)$$

Formula (5) is derived as the result of tedious algebra including the direct solving the system (9) and transformation of inequality (8) which exploit the properties for solution of system (9) and the explicit expressions for matrices $\widetilde{Y}_r, r = \overline{0,2}$. Theorem is proven.

Remark. It is worth to mention that the stability condition (5) does not depend on the distribution of customers among the servers. Condition becomes be more complicated in the case when some probabilities φ_m are allowed to be equal to zero, i.e., some servers are reserved only for the service of primary customers. However, the technique, which was exploited in the proof of the theorem above, can be used for that case successfully as well.

5. Stationary state probabilities

The stationary state distribution of the Markov chain $\xi_n, n \geq 1$ is calculated by means of direct application of the algorithm for asymptotically quasi-toeplitz Markov chains presented in [1]. Stationary distribution $\vec{p}_i, i \geq 0$ of considered queueing system is calculated then from (7).

References

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