Bootstrap Confidence Regions of 2-dimensional Vector-valued Process Capability Indices $\mathbf{C}_p$ and $\mathbf{C}_{pk}$

Byoung-Sun Park[1], Kyung-Hyun Nam[2] and Joong-Jae Cho[3*]

[1] Measurement & Quality Service Group, Korea Research Institute of Standards and Science
[2] Division of Economics, Kyonggi University
[3] Department of Information Statistics, Chungbuk National University

Abstract

In actual manufacturing industries, process capability indices(PCI) are used to determine whether a production process is capable of producing items within a specified specification limits. We study some vector-valued PCIs $\mathbf{C}_p = (C_{px}, C_{py})$ and $\mathbf{C}_{pk} = (C_{pkx}, C_{pky})$ in this article. We propose some asymptotic confidence regions of PCIs with bootstrapping and examine the performance of those asymptotic confidence regions under the assumption of bivariate normal distribution.

1. Introduction

In practicing quality control it is frequently of great interest to examine if the system is capable of performing its job successfully during its mission period. A variety of statistical techniques for analyzingprocess capability have been proposed and applied on a large scale to the manufacturing process adopted by many different kinds of industries since the early 1980's. To evaluate the degree of process capability of a system, it is necessary to define a quantitative measure that can explain the performance of the system. Out of such a necessity, several process capability indices have been introduced to assess whether a production process is capable of producing items which meet the requirements of the specified tolerance limits. As Flexible Manufacturing System is introduced, it is not difficult to monitor several characteristic values simultaneously. Also it is more reasonable to assume those values are associated under the multivariate distribution. So that some vector valued process capability indices should be used to control process system.

Kotz and Johnson(1993) extended capability process indices to multivariate case and Kocherlakota and Kocherlakota(1991) provided joint probability distribution function for $\hat{C}_{px}$ and $\hat{C}_{py}$ under the bivariate normal distribution. Recently, Park, Lee and Cho(2002) suggested asymptotic confidence regions for bivariate vector valued $\mathbf{C}_p$ and $\mathbf{C}_{pk}$.

In this article, we propose some efficient confidence regions for vector valued PCIs by using the bootstrapping technique. In section 2, asymptotic probability distributions for the plug-in estimators for the bivariate $\mathbf{C}_p = (C_{px}, C_{py})$ and $\mathbf{C}_{pk} = (C_{pkx}, C_{pky})$ are introduced and asymptotic normal confidence regions are proposed by computing the variance-covariance matrix under the bivariate normal distribution. In section 3, we derive the bootstrap consistency for PCIs and provide two types of bootstrap confidence regions. In section 4, we compare the coverage probabilities of approximate normal and two types of bootstrap confidence regions for $\mathbf{C}_p$ and $\mathbf{C}_{pk}$ by simulation study. It shows that approximate normal and the standard bootstrap techniques are better confidence regions than the approximate normal method when the underlying process distribution is unknown.

2. Vector-valued PCIs and Approximate Normal Confidence Regions

2.1 Asymptotic Distribution of Our Estimators

In this section, we study the asymptotic distribution of bivariate vector PCIs for $\mathbf{C}_p = (C_{px}, C_{py})$ and $\mathbf{C}_{pk} = (C_{pkx}, C_{pky})$ and provide easier and more useful confidence region.

$\mathbf{C}_p$ and $\mathbf{C}_{pk}$ are defined as follows :

$$\mathbf{C}_p = (C_{px}, C_{py}) = \left(\frac{d_x}{3\sigma_x}, \frac{d_y}{3\sigma_y}\right)$$

$$\mathbf{C}_{pk} = (C_{pkx}, C_{pky}) = \left(\frac{d_x - |\mu_x - M_x|}{3\sigma_x}, \frac{d_y - |\mu_y - M_y|}{3\sigma_y}\right)$$

where,

$$d_x = \frac{USL_x - LSL_x}{2},\ d_y = \frac{USL_y - LSL_y}{2},$$

$$M_x = \frac{USL_x + LSL_x}{2},\ M_y = \frac{USL_y + LSL_y}{2},$$

and USL_x, LSL_x, USL_y and LSL_y denote the upper and lower specification limits for the values of characteristics X and Y, to be measured.

Let's consider the estimators $\hat{\mathbf{C}}_p$ and $\hat{\mathbf{C}}_{pk}$ by the plug-in method.

$$\hat{\mathbf{C}}_p = (\hat{C}_{px}, \hat{C}_{py}) = \left(\frac{d_x}{3S_x}, \frac{d_y}{3S_y}\right)$$

$$\hat{\mathbf{C}}_{pk} = (\hat{C}_{pkx}, \hat{C}_{pky}) = \left(\frac{d_x - |\overline{X} - M_x|}{3S_x}, \frac{d_y - |\overline{Y} - M_y|}{3S_y}\right)$$

where,

$$\hat{\mu}_x = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,\ \hat{\mu}_y = \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i,$$

$$\hat{\sigma}_x^2 = S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \text{ and}$$

$$\hat{\sigma}_y^2 = S_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2.$$

We introduce some asymptotic results for our vector-valued PCIs $\mathbf{C}_p$ and $\mathbf{C}_{pk}$(Park et al.(2002)).

Lemma1

With two finite fourth central moments $\mu_{4x} = E(X - \mu_x)^4$ and $\mu_{4y} = E(Y - \mu_y)^4$, we obtain the following result as $n \to \infty$;

$$(Z_{1n}, Z_{2n}, Z_{3n}, Z_{4n}) = (\sqrt{n}(\overline{X} - \mu_x), \sqrt{n}(\overline{Y} - \mu_y), \sqrt{n}(S_x^2 - \sigma_x^2), \sqrt{n}(S_y^2 - \sigma_y^2)) \xrightarrow{d} (Z_1, Z_2, Z_3, Z_4) \overset{d}{=} MN(0, \Sigma_{4\times 4})$$

$$\mu_{ix} = E[(X - \mu_x)^i],\ \mu_{iy} = E[(Y - \mu_y)^i],\ i = 3, 4$$

$$\mu_{ixjy} = E[(X - \mu_x)^i (y - \mu_y)^j],\ i, j = 1, 2$$

Theorem1

With two finite fourth central moments $\mu_{4x} = E(X - \mu_x)^4$ and $\mu_{4y} = E(Y - \mu_y)^4$, we obtain the following result as $n \to \infty$;

$$\sqrt{n}(\hat{\mathbf{C}}_p - \mathbf{C}_p) = \sqrt{n}(\hat{\mathbf{C}}_{px} - \mathbf{C}_{px}, \hat{\mathbf{C}}_{py} - \mathbf{C}_{py}) \xrightarrow{d} \left(-\frac{d_x Z_3}{6\sigma_x^3}, -\frac{d_y Z_4}{6\sigma_y^4}\right) \overset{d}{=} MN(0, \mathbf{V}_p)$$

where $\tau_x^2 = \sigma_x^2 + (\mu_x - T_x)^2$, $\tau_y^2 = \sigma_y^2 + (\mu_y - T_y)^2$, $(Z_1, Z_2, Z_3, Z_4) \sim MN(0, \Sigma_{4\times 4})$ and

$$\Sigma_{4\times 4} = \begin{pmatrix} \sigma_x^2 & \mu_{1x1y} & \mu_{3x} & \mu_{1x2y} \\ & \sigma_y^2 & \mu_{2x1y} & \mu_{3y} \\ & symm. & \mu_{4x} - \sigma_x^4 & \mu_{2x2y} - \sigma_x^2\sigma_y^2 \\ & & & \mu_{4y} - \sigma_y^4 \end{pmatrix}.$$

$\mathbf{V}_p$ is obtained as follows.

$$\mathbf{V}_p = \begin{pmatrix} \frac{d_x^2}{36\sigma_x^6}(\mu_{4x} - \sigma_x^4) & \frac{d_x d_y}{36\sigma_x^3\sigma_y^3}(\mu_{2x2y} - \sigma_x^2\sigma_y^2) \\ & \frac{d_y^2}{36\sigma_y^6}(\mu_{4y} - \sigma_y^4) \end{pmatrix}$$

Similarly, we can obtain asymptotic distribution for vector-valued PCI $\mathbf{C}_{pk}$.

2.2 Approximate Normal Confidence Regions

If a bivariate random sample (X_1, Y_1), (X_2, Y_2), $\cdots$, (X_n, Y_n) is from any bivariate probability distribution with finite fourth central moments then $100(1-\alpha)\%$ approximate normal confidence region for bivariate process capability indices $\mathbf{C}_p$ and $\mathbf{C}_{pk}$ could be obtained.

2.2.1 $100(1-\alpha)\%$ Approximate Normal Confidence Region for $\mathbf{C}_p$

$100(1-\alpha)\%$ approximate normal confidence region for vector process capability index $\mathbf{C}_p$ would be

$$n(\hat{\mathbf{C}}_p - \mathbf{C}_p)'(\hat{\mathbf{V}}_p)^{-1}(\hat{\mathbf{C}}_p - \mathbf{C}_p) \le \chi^2_{(2;\alpha)}$$

where $\chi^2_{(2;\alpha)}$ denotes lower $100(1-\alpha)\%$ percentile of χ^2-distribution with 2 degrees of freedom and $\hat{\mathbf{V}}_p$ denotes the plug-in estimator for matrix $\mathbf{V}_p$. Each element of $\hat{\mathbf{V}}_p$ is obtained as follows.

$$\hat{\sigma}_{p11} = \widehat{Var}\left(-\frac{d_x Z_3}{6\sigma_x^3}\right) = \frac{d_x^2}{36\hat{\sigma}_x^6}(\hat{\mu}_{4x} - \hat{\sigma}_x^4) = \frac{d_x^2}{36\hat{\sigma}_x^6}\cdot 2\hat{\sigma}_x^4 = \frac{d_x^2}{18\hat{\sigma}_x^2}$$

$$\hat{\sigma}_{p22} = \widehat{Var}\left(-\frac{d_y Z_4}{6\sigma_y^3}\right) = \frac{d_y^2}{36\hat{\sigma}_y^6}(\hat{\mu}_{4y} - \hat{\sigma}_y^4) = \frac{d_y^2}{36\hat{\sigma}_y^6}\cdot 2\hat{\sigma}_y^4 = \frac{d_y^2}{18\hat{\sigma}_y^2}$$

$$\hat{\sigma}_{p12} = \hat{\sigma}_{p21} = \frac{d_x d_y}{36\hat{\sigma}_x^3\hat{\sigma}_y^3}\widehat{Cov}((X - \mu_x)^2, (Y - \mu_y)^2) = \frac{d_x d_y}{36\hat{\sigma}_x^3\hat{\sigma}_y^3}\cdot 2\hat{\rho}^2\hat{\sigma}_x^2\hat{\sigma}_y^2 = \frac{d_x d_y}{18\hat{\sigma}_x\hat{\sigma}_y}$$

2.2.2 $100(1-\alpha)\%$ Approximate Normal Confidence Region for $\mathbf{C}_{pk}$

$100(1-\alpha)\%$ approximate normal confidence region for vector process capability index $\mathbf{C}_{pk}$ would be

$$n(\hat{\mathbf{C}}_{pk} - \mathbf{C}_{pk})'(\hat{\mathbf{V}}_{pk})^{-1}(\hat{\mathbf{C}}_{pk} - \mathbf{C}_{pk}) \le \chi^2_{(2;\alpha)}$$

Each element of $\hat{\mathbf{V}}_{pk}$ is obtained as follows.

$$\hat{\sigma}_{pk11} = \widehat{Var}\left(-\frac{d_x Z_3}{6\sigma_x^3} - \frac{|Z_1|}{3\sigma_x}\right)$$
$$= \frac{d_x^2}{36\hat{\sigma}_x^6}(\hat{\mu}_{4x} - \hat{\sigma}_x^4) + \frac{\pi-2}{9\pi}$$
$$= \frac{d_x^2}{36\hat{\sigma}_x^6}\cdot 2\hat{\sigma}_x^4 + \frac{\pi-2}{9\pi}$$
$$= \frac{d_x^2}{18\hat{\sigma}_x^2} + \frac{\pi-2}{9\pi}$$

$$\hat{\sigma}_{pk22} = \widehat{Var}\left(\frac{(|M_x - \overline{Y}| - d_y)Z_4}{6\sigma_y^3} + \frac{Z_2}{3\sigma_y}\right)$$
$$= \frac{(|M_x - \overline{Y}| - d_y)^2}{36\hat{\sigma}_y^6}(\hat{\mu}_{4y} - \hat{\sigma}_y^4) + \frac{1}{9}$$
$$= \frac{(|M_x - \overline{Y}| - d_y)^2}{36\hat{\sigma}_y^6}\cdot 2\hat{\sigma}_y^4 + \frac{1}{9}$$
$$= \frac{(|M_x - \overline{Y}| - d_y)^2}{18\hat{\sigma}_y^2} + \frac{1}{9}$$

$$\hat{\sigma}_{pk12} = \hat{\sigma}_{pk21}$$
$$= -\frac{d_x(|M_x - \mu_y| - d_y)}{36\hat{\sigma}_x^3\hat{\sigma}_y^3} \times \widehat{Cov}((X-\mu_x)^2, (Y-\mu_y)^2)$$
$$- \frac{d_x}{18\hat{\sigma}_x^3\hat{\sigma}_y}\widehat{Cov}((X-\mu_x)^2, Y)$$
$$= -\frac{d_x(|M_x - \overline{Y}| - d_y)}{36\hat{\sigma}_x^3\hat{\sigma}_y^3}\cdot 2\hat{\rho}^2\hat{\sigma}_x^2\hat{\sigma}_y^2$$
$$= -\frac{d_x(|M_x - \overline{Y}| - d_y)\hat{\rho}^2}{18\hat{\sigma}_x\hat{\sigma}_y}$$

where,

$$\hat{\mu}_{4x} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^4,\ \hat{\mu}_{4y} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^4$$

and $\hat{\rho}$ denotes sample correlation coefficient.

3. Bootstrap Confidence Regions

The method of bootstrap was introduced by Efron(1979) as a nonparametric and computer intensive method. Such method allows calculation of confidence intervals for several parameters of interest without the usual assumption of normality. Hall(1988) provides excellent arguments explaining which type of bootstrapping technique should be used. Franklin and Wasserman(1992) propose three different types of bootstrap confidence intervals for C_p, C_{pk} and C_{pm}.

3.1 Bootstrap Asymptotic Distribution

Our bootstrap Algorithm is as follows.

[Step1] Obtain bootstrap sample of size m, (X_1^*, Y_1^*), (X_2^*, Y_2^*), $\cdots$, (X_m^*, Y_m^*) by replacement from the given random sample (X_1, Y_1), (X_2, Y_2), $\cdots$, (X_n, Y_n).

[Step2] Compute bootstrap sample mean $\overline{X}^*$, $\overline{Y}^*$and sample variance S_x^{*2}, S_y^{*2} by the following manner

$$\overline{X}^* = \frac{1}{m}\sum_{i=1}^{m}X_i^*,\ \overline{Y}^* = \frac{1}{m}\sum_{i=1}^{m}Y_i^*,$$
$$S_x^{*2} = \frac{1}{m-1}\sum_{i=1}^{m}(X_i^* - \overline{X}^*)^2,$$
$$S_y^{*2} = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i^* - \overline{Y}^*)^2$$

[Step3] Find vector-valued bootstrap estimators $\hat{\mathbf{C}}_p^*$ and $\hat{\mathbf{C}}_{pk}^*$.

$$\hat{\mathbf{C}}_p^* = (\hat{C}_{px}^*, \hat{C}_{py}^*) = \left(\frac{d_x}{3S_x^*}, \frac{d_y}{3S_y^*}\right)$$
$$\hat{\mathbf{C}}_{pk}^* = (\hat{C}_{pkx}^*, \hat{C}_{pky}^*)$$
$$= \left(\frac{d_x - |\overline{X}^* - M_x|}{3S_x^*}, \frac{d_y - |\overline{Y}^* - M_y|}{3S_y^*}\right)$$

First, we derive bootstrap asymptotic distribution and provide bootstrap consistency of our vector-valued PCIs $\mathbf{C}_p$.

Lemma2

With two finite fourth central moments $\mu_{4x} = E(X-\mu_x)^4$ and $\mu_{4y} = E(Y-\mu_y)^4$, we obtain the following result as $n \to \infty$;

$$(Z_{1m}^*, Z_{2m}^*, Z_{3m}^*, Z_{4m}^*)$$
$$= (\sqrt{m}(\overline{X}^* - \overline{X}), \sqrt{m}(\overline{Y}^* - \overline{Y}),$$
$$\sqrt{m}(S_x^{*2} - S_x^2), \sqrt{m}(S_y^{*2} - S_y^2)) | \chi_n$$
$$\xrightarrow{d} (Z_1, Z_2, Z_3, Z_4) \stackrel{d}{=} MN(0, \Sigma_{4\times 4})$$

where

$\chi_n = \{(X_1, Y_1), (X_2, Y_2), \cdots, ((X_n, Y_n)\}$ and

$$\Sigma_{4\times 4} = \begin{pmatrix} \sigma_x^2 & \mu_{1x1y} & \mu_{3x} & \mu_{1x2y} \\ & \sigma_y^2 & \mu_{2x1y} & \mu_{3y} \\ & symm. & \mu_{4x} - \sigma_x^4 & \mu_{2x2y} - \sigma_x^2\sigma_y^2 \\ & & & \mu_{4y} - \sigma_y^4 \end{pmatrix}.$$

Theorem2

With two finite fourth central moments $\mu_{4x} = E(X-\mu_x)^4$ and $\mu_{4y} = E(Y-\mu_y)^4$, we obtain the following result as $n \to \infty$;

$$\sqrt{n}(\hat{\mathbf{C}}_p^* - \hat{\mathbf{C}}_p) = \sqrt{n}(\hat{\mathbf{C}}_{px}^* - \hat{\mathbf{C}}_{px}, \hat{\mathbf{C}}_{py}^* - \hat{\mathbf{C}}_{py})$$
$$\xrightarrow{d} \left(-\frac{d_x Z_3}{6\sigma_x^3}, -\frac{d_y Z_4}{6\sigma_y^4}\right) \stackrel{d}{=} MN(0, \mathbf{V}_p)$$

where $\tau_x^2 = \sigma_x^2 + (\mu_x - T_x)^2$, $\tau_y^2 = \sigma_y^2 + (\mu_y - T_y)^2$, $(Z_1, Z_2, Z_3, Z_4) \sim MN(0, \Sigma_{4\times 4})$ and

$$\Sigma_{4\times 4}=\begin{pmatrix} \sigma_x^2 & \mu_{1x1y} & \mu_{3x} & \mu_{1x2y} \\ & \sigma_y^2 & \mu_{2x1y} & \mu_{3y} \\ & symm. & \mu_{4x}-\sigma_x^4 & \mu_{2x2y}-\sigma_x^2\sigma_y^2 \\ & & & \mu_{4y}-\sigma_y^4 \end{pmatrix}.$$

Also, we can provide bootstrap consistency for vector-valued PCI C_{pk}.

3.2 Bootstrap Confidence Region

Let C represent one of the vector valued PCI C_p and C_{pk}, $\hat{C}$ represent the its corresponding plug-in estimator and $\hat{C}^*$ be the bootstrap estimator of C from the bootstrap sample (X_1^*, Y_1^*), (X_2^*, Y_2^*), ⋯, (X_m^*, Y_m^*).

3.2.1 Standard Bootstrap(SB)

To build a confidence region, we find the sample mean vector $\bar{C}^*$ and variance-covariance matrix S_C^* from B bootstrap estimates for $\hat{C}^*=(\hat{C}_x^*, \hat{C}_y^*)$, $i=1,2,\cdots,B$ as follows.

$$\bar{C}^*=\begin{pmatrix}\bar{C}_x^*\\ \bar{C}_y^*\end{pmatrix}=\begin{pmatrix}\frac{1}{B}\sum_{i=1}^{B}\hat{C}_{xi}^*\\ \frac{1}{B}\sum_{i=1}^{B}\hat{C}_{yi}^*\end{pmatrix},\ S_C^*=\begin{pmatrix}S_{11}^* & S_{12}^*\\ S_{21}^* & S_{22}^*\end{pmatrix}$$

where,

$$S_{xy}^*=\frac{1}{B-1}\sum_{i=1}^{B}(\hat{C}_{xi}^*-\bar{C}_x^*)(\hat{C}_{yi}^*-\bar{C}_y^*).$$

Then $100(1-\alpha)\%$ SB confidence region for bivariate PCI C is

$$(\hat{C}-C)'(S_C^*)^{-1}(\hat{C}-C)\le \chi^2_{(2;\alpha)}$$

3.2.2 Studentized Bootstrap(STUD)

We could build a confidence region by locating the percentile point from the distribution of $m(\hat{C}^*-\hat{C})'(\hat{V}^*)^{-1}(\hat{C}^*-\hat{C})$.

Then $100(1-\alpha)\%$ STUD confidence region for bivariate PCI C is

$$n(\hat{C}-C)'(\hat{V})^{-1}(\hat{C}-C)\le \hat{y}_\alpha$$

where $\hat{y}_\alpha$ statisfies

$$\Pr\{m(\hat{C}^*-\hat{C})'(\hat{V}^*)^{-1}(\hat{C}^*-\hat{C})\le \hat{y}_\alpha | \chi_n\}=1-\alpha$$

and $\hat{V}$ and $\hat{V}^*$ represent the corresponding estimator and bootstrap estimator of variance-covariance matrix V for vector valued PCI C respectively.

4. Simulation Study

4.1 Simulation Procedures

To compare the performance of each method, we choose the bivariate normal distribution with parameter values as in the Table1. The original random sample and bootstrap sample size n and m are chosen to be equal for the convenience. We choose $n=30$, 60 and $\rho=$-0.9, -0.3, 0, 0.3, 0.9 under this design, a simulation procedure is done as following steps.

[Step1] Take $n(=30,60)$ bivariate samples from the bivariate normal distribution , then generate $B(=1000)$ bootstrap bivariate samples of size $m(=n)$ from the original random samples.

[Step2] Build 95% approximate normal(AN) and two types of bootstrap confidence region(SB, STUD) for the bivariate vector process capability indices C by using the sample from [Step1].

[Step3] Decide whether the true value of C is contained in the confidence region by [Step2].

This simple simulation was then replicated $N=1000$ times and thus a percentage of times the true value of the index is within the calculated interval could be obtained as a coverage probability.

Table1. Process parameter values in the simulation

	μ_x	μ_y	σ_x	σ_y	C	
$\mu_x=M_x$, $\mu_y=M_y$	50.0	100.0	3.0	3.0	(1.0,1.0)	C_p
			3.0	1.5	(1.0,2.0)	
			1.5	1.5	(2.0,2.0)	
$\mu_x<M_x$, $\mu_y<M_y$	45.5	95.5	3.0	3.0	(0.5,0.5)	C_{pk}
			3.0	1.0	(0.5,1.5)	
			1.0	1.0	(1.5,1.5)	
$\mu_x=M_x$, $\mu_y<M_y$	50.0	95.5	3.0	3.0	(1.0,0.5)	C_{pk}
			3.0	1.0	(1.0,1.5)	
			1.5	1.0	(2.0,1.5)	

4.2 Simulation Result

The result based on the simulation procedures discussed in section 4.1 are tabulated in Table2-4. In this section, we discuss our observations made from these tables in details.

First the 99% confidence interval for the 95% confidence region for 1000 simulations is obtained as (0.9333,0.967). Bold typed number in the table indicates that the coverage probability is beyond the scope of the interval. For the confidence region for C_p, STUD method performs poorly comparing with other two methods but as sample size increases there is no difference among three methods.

Most poor cases are located in the case of $\sigma_x>\sigma_y$. From the Table3, we find similar pattern as in the Table2. But there is somewhat different pattern in the Table4. All three cases for $\sigma_x=3.0$, $\sigma_y=1.0$ and

Table2. Coverage of confidence region for C_p

$(\mu_x, \mu_y, \sigma_x, \sigma_y)$	n	Method	-0.9	-0.3	0	0.3	0.9
(50.0,100.0,3.0,3.0)	30	AN	0.945	0.955	0.958	0.957	0.943
		SB	0.943	0.945	0.940	0.941	0.941
		STUD	0.939	0.946	0.938	0.943	0.945
	60	AN	0.939	0.951	0.948	0.949	0.945
		SB	0.939	0.945	0.945	0.942	0.949
		STUD	0.936	0.937	0.945	0.937	0.934
(50.0,100.0,3.0,1.5)	30	AN	**0.927**	**0.926**	**0.932**	**0.928**	**0.928**
		SB	**0.922**	**0.921**	0.935	**0.931**	**0.926**
		STUD	**0.917**	**0.914**	**0.918**	**0.922**	**0.931**
	60	AN	0.949	0.958	0.947	0.947	0.945
		SB	0.944	0.945	0.943	0.938	0.942
		STUD	0.941	0.944	0.942	0.937	0.941
(50.0,100.0,1.5,1.5)	30	AN	0.941	0.942	0.946	0.951	0.939
		SB	0.939	0.944	0.948	0.947	0.940
		STUD	0.943	**0.932**	**0.930**	0.941	0.937
	60	AN	0.941	0.955	0.952	0.958	0.948
		SB	0.941	0.949	0.954	0.947	0.939
		STUD	0.940	0.939	0.942	0.952	0.944

Table3. Coverage for $C_{pk}(\mu_x < M_x, \mu_y < M_y)$

$(\mu_x, \mu_y, \sigma_x, \sigma_y)$	n	Method	-0.9	-0.3	0	0.3	0.9
(45.5,95.5,3.0,3.0)	30	AN	0.955	0.962	0.962	0.951	0.955
		SB	0.939	0.945	0.956	0.952	0.951
		STUD	**0.932**	0.946	0.949	0.951	0.951
	60	AN	0.945	0.950	0.950	0.941	0.937
		SB	**0.932**	0.937	**0.932**	**0.932**	**0.930**
		STUD	0.938	0.935	0.938	**0.932**	0.935
(45.5,95.5,3.0,1.0)	30	AN	0.944	0.947	0.944	0.951	0.955
		SB	0.937	0.938	0.936	0.940	0.939
		STUD	0.934	**0.926**	**0.928**	**0.931**	0.944
	60	AN	0.955	0.965	0.958	0.956	0.959
		SB	0.947	0.957	0.960	0.949	0.944
		STUD	0.943	0.956	0.958	0.950	0.940
(45.5,95.5,1.0,1.0)	30	AN	0.952	0.946	0.945	0.953	0.939
		SB	0.940	0.941	0.942	0.949	0.933
		STUD	0.941	0.935	0.942	0.944	0.933
	60	AN	0.953	0.951	0.947	0.959	0.950
		SB	0.948	0.951	0.953	0.956	0.942
		STUD	0.946	0.945	0.947	0.952	0.947

Table4. Coverage for $C_{pk}(\mu_x = M_x, \mu_y < M_y)$

$(\mu_x, \mu_y, \sigma_x, \sigma_y)$	n	Method	-0.9	-0.3	0	0.3	0.9
(50.0,95.5,3.0,3.0)	30	AN	0.934	0.961	0.959	0.958	**0.927**
		SB	0.934	0.933	0.945	0.942	0.947
		STUD	**0.924**	0.937	0.943	0.934	**0.930**
	60	AN	**0.927**	0.948	0.949	0.944	**0.920**
		SB	0.936	0.941	0.938	0.940	0.936
		STUD	0.940	0.941	0.938	0.938	0.935
(50.0,95.5,3.0,1.0)	30	AN	**0.907**	**0.926**	**0.930**	**0.929**	**0.915**
		SB	**0.912**	**0.916**	**0.919**	**0.923**	**0.916**
		STUD	**0.916**	**0.910**	**0.916**	**0.913**	**0.914**
	60	AN	**0.930**	0.939	0.939	**0.932**	**0.922**
		SB	0.933	0.939	0.936	**0.923**	0.937
		STUD	**0.926**	0.934	**0.930**	**0.927**	0.933
(50.0,95.5,1.0,1.0)	30	AN	0.938	0.941	0.950	0.951	0.940
		SB	0.933	**0.931**	0.942	0.943	0.940
		STUD	0.933	**0.930**	0.934	0.938	**0.932**
	60	AN	0.945	0.943	0.948	0.950	0.939
		SB	0.935	0.943	0.948	0.946	0.937
		STUD	0.936	**0.931**	0.942	0.944	0.939

$n = 30$ perform poorly but as n increases to 60 this undesirable property changes. In all cases AN method yields higher coverage probability than other two methods.

5. Conclusions

For each index, regardless of the underlying distribution or the sample size, the coverage probability of AN method is higher than the other two. There is another tendency of better performing as we increase the number n of sample size. In conclusion, AN and SB methods are better than STUD method. It is recommended that the SB method should be used when there is no informations available regarding underlying distribution which is the most case in the practice.

References

[1] Efron, B., Bootstrap Methods:Another Look at the jackknife, *Annals of Statistics,* 7, 1-26, 1979.

[2] Franklin, L.A. and Wasserman, G.S., Bootstrap Lower Confidence Interval Limits for Capability Indices, *Journal of Quality Technology,* 24, 196-210, 1992.

[3] Hall, P., Theoretical Comparison of Bootstrap Confidence Intervals. *Annals of Statistics,* 16, 927-953, 1988.

[4] Kocherlakota, S. and Kocherlakota, K., Process Capability Indices : Bivariate Normal Distribution, *Communication in Statistics : Theory and Methods,* 20, 2529-2547, 1991.

[5] Kotz, S. and Johnson, N. L., *Process Capability Indices,* Chapman & Hall, 1993.

[6] Park, B. S., Lee, C. H. and Cho, J. J., On the Confidence Region of Vector-valued Process Capability Indices Cp and Cpk, Journal of the Korean Society for Quality Management, 30, 4, 44-57, 2002.