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Estimation for the scale parameter of Weibull Distribution Based on Multiply Censored Samples

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Abstract

We consider the problem of estimating the scale parameter of the Weibull distribution based on multiply Type-II censord samples. We propose some estimators by using the approximate maximum likelihood estimation method. The proposed estimators are compared in the sense of the mean squared error.

Keywords : Multiply type-II censored samples, Weibull distribution, Approximate maximum likelihood estimators

1. Introduction

The probability density function of the random variable X having a Weibull distribution with the scale parameter β and the shape parameter δ is given by

$$f(x)=\frac{\delta}{\beta^{\delta}}x^{\delta-1}\exp\left\{-\left(\frac{x}{\beta}\right)^{\delta}\right\},\qquad \beta>0,\quad \delta>0,\quad x>0 \tag{1.1}$$

Estimations for the parameters in the Weibull distribution has been studied for censored samples. In most cases of censored samples, Estimators of parameters may not be obtained by the maximum likelihood method. The approximate maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution. Estimation based on censored samples have been studied by many authors. Kang et al. (2001) obtained the approximate maximum likelihood estimators (AMLEs) for the parameters in the three-parameter Weibull distribution.

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For multiply Type-II censored samples, Balakrishnan et al. (1995) derived the estimators for the location and scale parameters of the extreme value distribution. Fei and Kong (1995) compared the mean squared errors (MSEs) of the maximum likelihood estimators (MLEs), AMLEs, beat linear unbiased estimators (BLUEs) of the parameters in extreme value distribution.

In this paper, we derive the AMLEs for the scale parameter of the Weibull distribution based on multiply Type-II censored samples. We also compare the proposed estimators in the sense of MSE various censored samples.

2. Estimator of the scale parameter

Consider the Weibull distribution with cumulative density function (cdf)

$$F(x;\beta,\delta)=1-\exp\left\{-\left(\frac{x}{\beta}\right)^{\delta}\right\} \tag{2.1}$$

We assume that δ is known.

Let us assume that the following multiply Type-II censored sample from a sample of size n

$$X_{a_1:n}<X_{a_2:n}<X_{a_3:n}< \cdots <X_{a_s:n} \tag{2.2}$$

where $1\le a_1<a_2< \cdots <a_s\le n$

$$a_0=0,\ a_{s+1}=n+1\ F(x_{a_0})=0,\ F(x_{a_{s+1}})=1.$$

The likelihood function based on the multiply Type-II censored sample (2.2) can be written as

$$L=n!\prod_{j=1}^{s}f(x_{a_j})\prod_{j=1}^{s+1}\frac{[F(x_{a_j})-F(x_{a_{j-1}})]^{a_j-a_{j-1}-1}}{(a_j-a_{j-1}-1)!}. \tag{2.3}$$

By putting $Z_{i:n}=\frac{X_{i:n}}{\beta}$, the likelihood function can be rewritten as

$$L=n!\prod_{j=1}^{s+1}\frac{[F(z_{a_j})-F(z_{a_{j-1}})]^{a_j-a_{j-1}-1}}{(a_j-a_{j-1}-1)!}\prod_{j=1}^{s}\frac{f(z_{a_j})}{\beta} \tag{2.4}$$

where $f(z)=\delta z^{\delta-1}\exp(-z^{\delta})$ and $F(z)=1-\exp(-z^{\delta})$ are the pdf and the cdf of the standard Weibull distribution, respectively.

We have the log-likelihood function to be

$$\begin{aligned}\ln L &= \ln\frac{n!}{\prod_{j=1}^{s+1}(a_j-a_{j-1}-1)} - s\ln\beta + (a_1-1)\ln F(Z_{a_1:n}) \\ &+ \sum_{j=2}^{s}(a_j-a_{j-1}-1)\ln\{F(z_{a_j:n})-F(z_{a_{j-1}:n})\} + (n-a_s).\end{aligned} \tag{2.5}$$

Then the likelihood equation for β is given by

$$\begin{aligned}\frac{\partial \ln L}{\partial\beta} &= -\frac{1}{\beta}[s+(a_1-1)\frac{f(z_{a_1})}{F(z_{a_1})}z_{a_1} - (n-a_s)\frac{f(z_{a_s})}{1-F(z_{a_s})}z_{a_s} \\ &+ \sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(z_{a_j})z_{a_j}-f(z_{a_{j-1}})z_{a_{j-1}}}{F(z_{a_j})-F(z_{a_{j-1}})} + \sum_{j=1}^{s}\frac{f'(z_{a_j})}{f(z_{a_j})}z_{a_j} \\ &= 0\end{aligned} \tag{2.6}$$

The equation (2.6) can't be solved explicitly for β. But we may expand the functions $\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n}$, $\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})}z_{a_s:n}$, $\frac{f'(z_{a_j:n})}{f(z_{a_j:n})}z_{a_j:n}$, $\frac{f(z_{a_j:n})z_{a_j:n}-f(z_{a_{j-1}:n})z_{a_{j-1}:n}}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}$ in Taylor series around the points ξ_{a_1}, ξ_{a_s}, ξ_{a_j}, $(\xi_{a_j}, \xi_{a_{j-1}})$.

Let $p_i=\frac{i}{n+1}$, $F^{-1}(p_i)=[-\ln(1-p_i)]^{\frac{1}{\delta}}=\xi_i$,, we may approximate these functions as

$$\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n} \simeq \alpha_1+\gamma_1 z_{a_1:n} \tag{2.7}$$

$$\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})}z_{a_s:n} \simeq \alpha_s+\gamma_s z_{a_s:n} \tag{2.8}$$

$$\frac{f'(z_{a_j:n})}{f(z_{a_j:n})}z_{a_j:n} \simeq \alpha_j+\gamma_j z_{a_j:n} \tag{2.9}$$

$$\frac{f(z_{a_j:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}z_{a_j:n} \simeq \alpha_{1j}+\gamma_{1j}z_{a_j:n}+x_{1j}z_{a_{j-1}:n} \tag{2.10}$$

and

$$\frac{f(z_{a_j:n})}{F(z_{a_{j-1}:n})-F(z_{a_j:n})}z_{a_{j-1}:n} \simeq \alpha_{2j}+\gamma_{2j}z_{a_j:n}+x_{2j}z_{a_{j-1}:n}, \tag{2.11}$$

where

$$\alpha_1 = -C_1\xi_{a_1}$$

$$\gamma_1 = C_1+\frac{f(\xi_{a_1})}{p_{a_1}}$$

$$C_1=\frac{p_{a_1}(\delta-1-\delta\xi^{\delta}_{a_1})-\xi_{a_1}f(\xi_{a_1})}{(p_{a_1})^2}f(\xi_{a_1})$$

$$\alpha_s=-C_s\xi_{a_s}$$

$$\gamma_s=C_s+\frac{f(\xi_{a_s})}{1-p_{a_s}}$$

$$C_s=\frac{(1-p_{a_s})(\delta-1-\delta\xi^{\delta}_{a_s})-\xi_{a_s}f(\xi_{a_s})}{(1-p_{a_s})^2}f(\xi_{a_s})$$

$$\alpha_j=(\delta-1)(1+\delta\,\xi^{\delta}{}_{a_j})$$

$$\gamma_j=-\delta^2\,\xi_{a_j}{}^{\delta-1}$$

$$\alpha_{1j}=-\frac{(\delta-1-\delta\;\xi^{\delta}{}_{a_j})\xi_{a_j}f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}+\frac{\xi_{a_{j-1}}f(\xi_{a_{j-1}})-\xi_{a_j}f(\xi_{a_j})}{[p_{a_j}-p_{a_{j-1}}]^2}\xi_{a_j}f(\xi_{a_j})$$

$$\gamma_{1j}=\frac{\delta(1-\;\xi^{\delta}{}_{a_j})}{p_{a_j}-p_{a_{j-1}}}f(\xi_{a_j})-\frac{f^2(\xi_{a_j})\xi_{a_j}}{[p_{a_j}-p_{a_{j-1}}]^2}$$

$$\varkappa_{1j}=\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})\xi_{a_j}}{[p_{a_j}-p_{a_{j-1}}]^2}$$

$$\alpha_{2j}=-\frac{(\delta-1-\delta\;\xi^{\delta}{}_{a_{j-1}})\xi_{a_{j-1}}f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}-\frac{\xi_{a_{j-1}}f(\xi_{a_{j-1}})-\xi_{a_j}f(\xi_{a_j})}{[p_{a_j}-p_{a_{j-1}}]^2}\xi_{a_{j-1}}f(\xi_{a_{j-1}})$$

$$\gamma_{2j}=-\frac{f(\xi_{a_{j-1}})f(\xi_{a_j})\xi_{a_{j-1}}}{[p_{a_j}-p_{a_{j-1}}]^2}$$

and

$$\varkappa_{2j}=\frac{\delta(1-\;\xi^{\delta}{}_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}f(\xi_{a_{j-1}})-\frac{f^2(\xi_{a_{j-1}})\xi_{a_{j-1}}}{[p_{a_j}-p_{a_{j-1}}]^2}.$$

By substituting (2.7), (2.8), (2.9), (2.10), and (2.11) into (2.6), we obtain the approximate likelihood equation for β as follows

$$\begin{aligned}\frac{\partial\ln L}{\partial\beta}\simeq\; & -\frac{1}{\beta}[s+(a_1-1)(\alpha_1+\gamma_1)z_{a_1}-(n-a_s)(\alpha_s+\gamma_s)z_{a_s}\\ & +\sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\alpha_{1j}-\alpha_{2j})+(\gamma_{1j}-\gamma_{2j})z_{a_j}+(\varkappa_{1j}-\varkappa_{2j})z_{a_{j-1}}]\\ & +\sum_{j=1}^{s}(\alpha_j+\gamma_j)z_{a_j}=0.\end{aligned}\tag{2.12}$$

Upon solving the equation for β, we derive the estimator of β as follows;

$$\widehat{\beta_w}=\frac{B}{A}\tag{2.13}$$

where

$$A= -(a_1-1)\gamma_1 X_{a_1:n}+(n-a_s)\gamma_s X_{a_s:n}$$
$$-\sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\gamma_{1j}-\gamma_{2j})X_{a_j:n}+(\kappa_{1j}-\kappa_{2j})X_{a_{j-1}:n}]-\sum_{j=1}^{s}\gamma_j X_{a_j:n}$$

$$B= s+(a_1-1)\alpha_1-(n-a_s)\alpha_s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}-\alpha_{2j})+\sum_{j=1}^{s}\alpha_j$$

3. Estimation based on the extreme value distribution

Let X be a random variable with pdf (1.1). Then it is easy to see that the $Y=\ln X$ has its density function to be

$$f(y;\mu,\delta)=\frac{1}{\sigma}e^{\frac{(y-\mu)}{\sigma}}\exp\left\{-e^{\frac{(y-\mu)}{\sigma}}\right\} \tag{3.1}$$

where $\sigma=\frac{1}{\delta}$, $\mu=\ln\beta$ and its cumulative distribution function to be

$$F(y;\mu,\delta)=1-\exp\left\{-e^{\frac{(y-\mu)}{\sigma}}\right\}. \tag{3.2}$$

That is, Y has the extreme value distribution with the location parameter μ and the scale parameter σ.

By putting $Z_{i:n}=\frac{Y_{i:n}-\mu}{\sigma}$, the likelihood function can be rewritten as

$$L = n!\prod_{j=1}^{s+1}\frac{[F(z_{a_j})-F(z_{a_{j-1}})]^{a_j-a_{j-1}-1}}{(a_j-a_{j-1}-1)!}\prod_{j=1}^{s}\frac{f(z_{a_j})}{\sigma} \tag{3.3}$$

where $f(z)=e^z\exp(-e^z)$ and $F(z)=1-\exp(-e^z)$ are the pdf and the cdf of the standard extreme value distribution.

Then, the likelihood equation for μ is given by

$$\begin{aligned}\frac{\partial\ln L}{\partial\mu} &=[(a_1-1)\frac{f(z_{a_1})}{F(z_{a_1})}(-\frac{1}{\sigma})-(n-a_s)\frac{f(z_{a_s})}{1-F(z_{a_s})}(-\frac{1}{\sigma})\\ &+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(z_{a_j})(-\frac{1}{\sigma})-f(z_{a_{j-1}})(-\frac{1}{\sigma})}{F(z_{a_j})-F(z_{a_{j-1}})}\\ &+\sum_{j=1}^{s}\frac{f(z_{a_j})}{f(z_{a_j})}(-\frac{1}{\sigma})=0.\end{aligned} \tag{3.4}$$

We may expand the functions

$$\frac{f(z_{a_1:n})}{F(z_{a_1:n})},\ \frac{f(z_{a_s:n})}{1-F(z_{a_s:n})},\ \frac{f'(z_{a_j:n})}{f(z_{a_j:n})},\ \frac{f(z_{a_j:n})-f(z_{a_{j-1}:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}$$

in Taylor series around the points ξ_{a_1}, ξ_{a_s}, ξ_{a_j}, $(\xi_{a_j},\xi_{a_{j-1}})$ as follows;

$$\frac{f(z_{a_1:n})}{F(z_{a_1:n})}\simeq\alpha_{E1}+\gamma_{E1}z_{a_1:n} \tag{3.5}$$

$$\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})}\simeq\alpha_{Es}+\gamma_{Es}z_{a_s:n} \tag{3.6}$$

$$\frac{f'(z_{a_j:n})}{f(z_{a_j:n})}\simeq\alpha_{Ej}+\gamma_{Ej}z_{a_j:n} \tag{3.7}$$

$$\frac{f(z_{a_j:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}\simeq\alpha_{E1j}+\gamma_{E1j}z_{a_j:n}+\chi_{E1j}z_{a_{j-1}:n} \tag{3.8}$$

and

$$\frac{f(z_{a_j:n})}{F(z_{a_{j-1}:n})-F(z_{a_j:n})}\simeq\alpha_{E2j}+\gamma_{E2j}z_{a_j:n}+\chi_{E2j}z_{a_{j-1}:n}, \tag{3.9}$$

where

$$\alpha_{E1}=\frac{f(\xi_{a_1})}{p_{a_1}}-A_1\xi_{a_1}$$

$$\gamma_{E1}=A_1$$

$$A_1=\frac{(1-e^{\xi_{a_1}})-f(\xi_{a_1})}{(p_{a_1})^2}f(\xi_{a_1})$$

$$\alpha_{Es}=\frac{f(\xi_{a_s})}{1-p_{a_s}}-A_s\xi_{a_s}$$

$$\gamma_{Es}=A_s$$

$$A_s=\frac{(1-e^{\xi_{a_s}})(1-p_{a_s})+f(\xi_{a_s})}{(1-p_{a_s})^2}f(\xi_{a_s})$$

$$\alpha_{Ej}=1-e^{\xi_{a_j}}+e^{\xi_{a_j}}\xi_{a_j}$$

$$\gamma_{Ej}=-e^{\xi_{a_j}}$$

$$\alpha_{E1j}=\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}-A_{1j}\xi_{a_j}-B_{1j}\xi_{a_{j-1}}$$

$$\gamma_{E1j}=A_{1j}$$

$$\chi_{E1j}=B_{1j}\ \chi_{1j}=\frac{f(\xi_{a_{j-1}})f(\xi_{a_j})\xi_{a_j}}{[p_{a_j}-p_{a_{j-1}}]^2}$$

$$A_{1j} = \frac{(1-e^{\xi_{a_j}})(p_{a_j}-p_{a_{j-1}})+f(\xi_{a_j})}{(p_{a_j}-p_{a_{j_1}})^2} f(\xi_{a_j})$$

$$B_{1j} = \frac{f(\xi_{a_j})f(\xi_{a_{j-1}})\xi_{a_j}}{[p_{a_j}-p_{a_{j-1}}]^2}$$

$$\alpha_{E2j} = \frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}} + B_{1j}\xi_{a_j} - B_{2j}\xi_{a_{j-1}}$$

and

$$x_{E2j} = \frac{(1-e^{\xi_{a_{j-1}}})[p_{a_j}-p_{a_{j-1}}]+f(\xi_{a_{j-1}})}{[p_{a_j}-p_{a_{j-1}}]^2} f(\xi_{a_{j-1}}).$$

By substituting (3.5), (3.6), (3.7), (3.8), and (3.9) into (3.4), we can obtain the estimatior of μ as follows;

$$\hat{\mu} = \frac{B_E}{A_E}$$

where

$$A_E = (a_1-1)\gamma_{E1} - (n-a_s)\gamma_{Es} + \sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\gamma_{E1j}-\gamma_{E2j})+(x_{E1j}-x_{E2j})] + \sum_{j=1}^{s}\gamma_{Ej}$$

$$\begin{aligned} B_E &= -(a_1-1)(\alpha_{E1}\sigma+\gamma_{E1}Y_{a_1:n}) + (n-a_s)(\alpha_{Es}\sigma+\gamma_{Es}Y_{a_s:n}) \\ &+ \sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\alpha_{E1j}-\alpha_{E2j})\sigma-(\gamma_{E1j}-\gamma_{E2j})Y_{a_j:n}+(x_{E1j}-x_{E2j})Y_{a_{j-1}:n}] \\ &- \sum_{j=1}^{s}(\sigma_{Ej}\sigma+\gamma_{Ej}X_{a_j:n}). \end{aligned}$$

Since $\mu = \ln\beta$, we can obtain the approximate maximum likelihood estimator of β as follows.

$$\widehat{\beta_E} = e^{\widehat{\mu}} \tag{3.10}$$

We simulate the MSEs for the proposed estimators of β in the Weibull distribution and extreme value distribution for various censored samples. The simulation procedure is repeated 10,000 times in multiply censored sample with $\beta = 1,2$ and sample size $n = 5,10$. From Table 1, the MSEs of the $\widehat{\beta_w}$ is smaller than the MSEs of the $\widehat{\beta_E}$ in the $\delta = 2$.

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Table 1 : the mean squared errors for the proposed estimators of the scale parameter β

n	k	a_j	MSE			
			$\beta=1,\ \delta=2$		$\beta=2,\ \delta=2$	
			$\widehat{\beta_w}$	$\widehat{\beta_E}$	$\widehat{\beta_w}$	$\widehat{\beta_E}$
5	0	1 2 3 4 5	0.049	0.050	0.194	0.201
	1	1 2 3 4	0.060	0.061	0.240	0.243
		2 3 4 5	0.049	0.050	0.195	0.200
		1 3 4 5	0.049	0.051	0.196	0.202
	2	2 3 4	0.060	0.061	0.242	0.243
		2 3 5	0.050	0.051	0.202	0.205
		1 3 5	0.051	0.052	0.202	0.207
10	0	1 2 3 4 5 6 7 8 9 10	0.025	0.025	0.099	0.101
	1	2 3 4 5 6 7 8 9 10	0.025	0.025	0.099	0.101
		1 2 3 4 5 6 7 8 9	0.027	0.028	0.109	0.110
		1 2 3 4 6 7 8 9 10	0.025	0.025	0.099	0.101
		1 2 3 5 6 7 8 9 10	0.025	0.025	0.099	0.101
	3	4 5 6 7 8 9 10	0.025	0.025	0.099	0.101
		1 2 3 4 5 6 7	0.035	0.035	0.139	0.140
		3 4 5 6 7 8 9	0.027	0.028	0.109	0.110
		2 3 4 5 6 7 8	0.031	0.031	0.122	0.123
		2 3 4 6 8 9 10	0.025	0.025	0.099	0.101
		3 4 5 7 8 9 10	0.025	0.025	0.099	0.101
	5	3 4 5 6 7	0.035	0.035	0.140	0.140
		2 3 4 5 6	0.040	0.041	0.162	0.162
		4 5 6 7 8	0.031	0.031	0.123	0.123
		2 4 6 8 10	0.025	0.026	0.101	0.102
		1 3 5 7 9	0.028	0.028	0.110	0.112
		2 3 4 7 8	0.031	0.031	0.123	0.124
		3 4 6 8 9	0.027	0.028	0.110	0.111

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