Better Bounds in Networks based on Randomly-Wired Expander

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Abstract

Linear size expanders have been studied in many fields for the practical use, which is possibility to connect large numbers of device chips in parallel communication systems. One major limitation on the efficiency of parallel computer designs has been the prohibitively high cost of parallel communication between processors and memories. Linear size expanders can be used to construct theoretically optimal interconnection networks. In current, the defined constructions have large constant factors, thus rendering them impractical for reasonable sized networks. This paper presents an improvement on constructing concentrators using an $(n,k,2rs/(r^2-s^2))$ expander, which realizes the reduction of the size in a superconcentrator by a constant factor.

Key words : Concentrator, Expander, Superconcentrator, Interconnection network

1 Introduction

One of the most important issues in parallel high computing system is the communication between processors in a message-passing architecture, and between processors and memory in a shared global memory machine. In order to achieve high-bandwidth parallel communication, it is necessary to be able to move information in parallel along separate, or disjoint, communication pathways. Interconnection networks have limitations in either the complexity of the routing computation[8], or in the time required to move the information.[2,9]

According to this thing, many a family of linear sized expanders, like concentrators and superconcentrators having $O(n)$ hardware have been developed during the past a couple of decades. Pippenger[10], showed how to build such a superconcentrator, by using a more primitive two-stage structure called a concentrator.

Pippenger gave a new version of the nonconstructive existence theorem and a simple recursive construction with 39$n$edges with the good and the bad one argument. However, the explicit construction is still needed for many applications. Chung[3] improved it to 261.5n edges and Alon and Milman[1] to 175n edges again. Since that, the construction different from that of Gabber and Galil was made with 248n edges by Jimbo and Maruoka[5]. Recently Leighton[6] proposed a fast routing algorithm can be used to compute the routing of the links through splitter network. Thus the remaining task is to determine the explicit construction of a practical sized concentrator.

In order to develop the construction of explicit linear sized concentrators and superconcentrators, Margulis[7] was the first to describe a family of linear size expanders in bipartite graph $\hat{G}_n$ and proved $|\Gamma_\lambda| \geq [1+d(1-|X|/n)]|X|$ for any subset X of input vertices with $|X| \leq n/2$. It is an (n,r,d) expander having n input vertices, n output vertices and at most kn edges. A construction of similar method is built by Gabber and Galil[4] with 404n and 271.8n edges in a family of explicit graph G_n, which are $(n,5,d)$ and $(n,7,2d)$ expanders respectively, where d is $(2-\sqrt{3})/4$.

Now, this paper proposes that an improvement of concentration coefficient can reduce the size of the resulting concentrator built from any given expander by a new constant factor as trying to find out the closer value to d. The basic expander is described in section 2. Through our improvement such an $(n,k,2rs/(r^2-s^2))$ expander from Gabber and Galil's, we will show in the section 3 and section 4 that it improve the density of the construction that Gabber and Galil constructed and it is the principal contribution of this paper. The section 5 shows how many density can be improved with applying it to them. Finally, conclusions are given in section 6.

2. Preliminary

The basic expander defined by Gabber and Galil consists of a set of $n$inputs, where $n=m^2$, m is any integer, and an equal number of outputs. The inputs are connected to the outputs by a set of five permutations, which shift each row right or left several columns, with wrap-around. Each successive row is shifted one more places until the middle row is shifted back onto itself, and so on. Identity and column increment permutations are included, and a separate set of similar permutations shifts the columns by rows. These permutations are described by the following functions, where addition is mod m:

$$\sigma_0(x,y) = (x,y)$$
$$\sigma_1(x,y) = (x, x+y)$$
$$\sigma_2(x,y) = (x, x+y+1)$$
$$\sigma_3(x,y) = (x+y, y)$$
$$\sigma_4(x,y) = (x+y+1, y)$$

First of all, a concentrator is defined to show how to build a superconcentrator through it. An (n,θ,k) concentrator is a directed acyclic graph with n input vertices, θn output vertices($\theta < 1$), and at most kn connections from the inputs to the outputs, and it has property that, for every subset of inputs X such that $|X| \le n/2$, there exist at least X flows connected from input to output vertices. In order to build such concentrators explicitly from expanders, an (n,k,d) expander, as used through this paper, is a two-stage network with n inputs, n outputs, with each inputs connected by links to kn outputs. The links are chosen in such a way that, for every set of inputs X, such that $|X| \le n/2$, the set of outputs Γ_X which are connected by links to X, observe the rule that:

$$|\Gamma_X| \ge \left[1 + d\left(1 - \frac{|X|}{n}\right)\right]|X| \qquad (1)$$

A superconcentrator is a directed acyclic graph with n inputs and n outputs such that, for every set of inputs and the same number set of outputs, there exists a set of flows connected from the inputs to the outputs.

3. Expander-Based Concentrator

A concentrator with n inputs and $nr/(r+s)$ outputs can be constructed from the expander defined in the last section. It is composed of two parts, called *part A* and *part B*, as shown in Fig.1. Thus, the inputs n are divided into two parts that the one is $nr/(r+s)$, the other is $ns/(r+s)$, respectively:

$$n = \frac{nr}{r+s} + \frac{ns}{r+s} \qquad (2)$$

, where n, r, s, and r/s are natural number and $nr/(r+s)$ is square number.

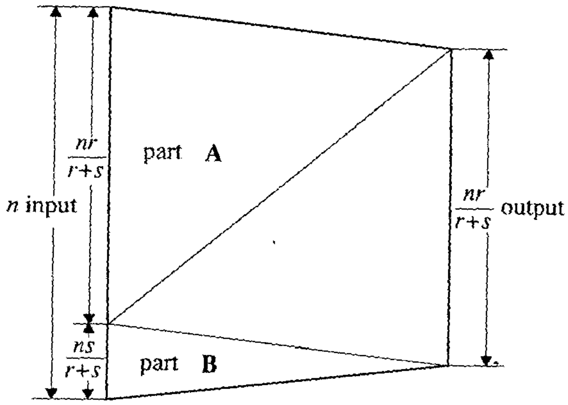

Fig.1.The concentrator composed of two different sized parts

Each part can be connected to output with different pattern. First, the *part A* can be connected to $nr/(r+s)$ outputs with an $[nr/(r+s), k, 2rs/(r^2-s^2)]$ expander as shown in Fig.2.(a). Next, each input in *part B* has r/s disjointed connections of $nr/(r+s)$ outputs as shown in Fig.2(b). In the next section, it is shown that this structure is an $(n, r/(r+s), k)$ concentrator for a specific range of values

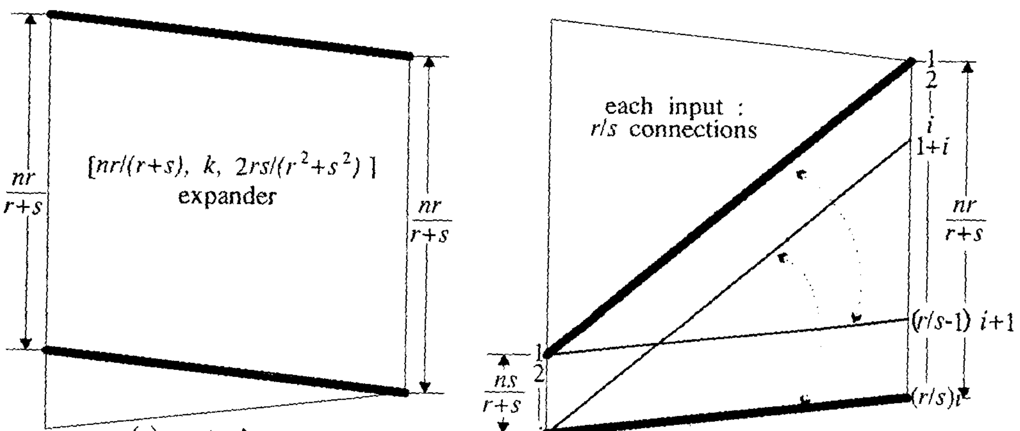

Fig.2.(a)The connections of part A (b)The connections of part B

of the concentration coefficient r/s.

4. The Size of Concentrator

In order to prove that the network defined in last section is an $(n, r/(r+s), k)$ concentrator, we have only to show that for every selection of input set X, the number of outputs to which it connects is at least equal to $|X|$. *An* (n,θ,k,α,c) *bounded strong concentrator* is a bipartite graph that there are $n$inputs, θn outputs, and at most kn edges. This means that for all subset of inputs with $|X| \le \alpha n$, $|\Gamma_X| \ge c|X|$, where $|\Gamma_X|$ denotes the set that is *adjacent* to set in X. The word *strong* and c can be left out when $c$is one. If the defined above is applied for *part A and part B*:

$$[nr/(r+s)]\cdot k + [ns/(r+s)]\cdot(r/s) = [(k+1)\cdot r/(r+s)]\cdot n$$

Let

$$k' = [(k+1)\cdot r/(r+s)]. \qquad (3)$$

Then, That an $(n, r/(r+s), k', 1/2)$ is bounded concentrator is bipartite graph is followed by theorem 1.

Theorem 1 : An $(n, r/(r+s), k)$ concentrator can be constructed from an (n,k,d) expander for every subset inputs X.

Proof: An (n,k,d) expander has the property of ineq.(1). In order to prove that $|\Gamma_X| \ge |X|$ for all cases, we can distinguish between *part A* and *part B*. And let a and b be the set of inputs in each part as shown in Fig.3. Let X_a and X_b be any subset of $X \cap a$ and $X \cap b$, respectively. First, if we consider the case of $|X_b| \ge (s/r)|X|$, $|\Gamma_X|$ is greater than or equal to $|X|$ since every input in part B has r/sconnections in output.

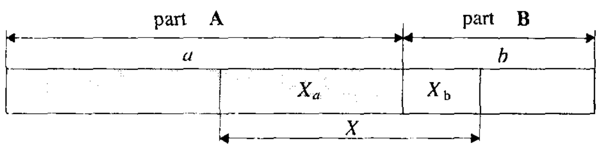

Fig.3. The set organization of inputs

Second, we use the definition of $(nr/(r+s), k, 2rs/(r^2-s^2))$ expander in the case of $|X_a| \ge [(r-s)/r]|X|$. By the definition of an $(nr/(r+s), k, 2rs/(r^2-s^2))$ expander, $|X_a|$ and $|X|$can have the scope such that $(r-s)/r|X| \le |X_a| \le |X|$ and $|X| \le \tilde{n}/2$, where $\tilde{n}$ is $nr/(r+s)$. Also, we can have the property as follows:

$$|\Gamma_{X_a}| \ge [1 + d(1 - |X_a|/\tilde{n})]|X_a| \qquad (4)$$

To find out the condition of $|\Gamma_{X_a}|_{\min}$ from ineq.(4)

$$\frac{d|\Gamma_{X_a}|}{d|X_a|} \ge -\frac{2d_0}{\tilde{n}}\left(|X_a| - \frac{\tilde{n}}{2}\cdot\frac{1+d_0}{d_0}\right) \qquad (5)$$

From ineq.(5), $|\Gamma_{X_a}|$ has maximum value when $|X_a| = (\tilde{n}/2)\cdot(1+d)/d$, but this value is not available since $|X_a|$ is at most $\tilde{n}/2, (|X_a| \le \tilde{n}/2)$. The number, $(\tilde{n}/2)\cdot(1+d)/d$, is always greater than $\tilde{n}/2$. Regardless of maximum value, $|\Gamma_{X_a}|_{\min}$ has minimum value when $|X_a|$ is $[(r-s)/r]|X|$ since it is increasing function in the given range as shown in Fig.4. Therefore:

$$|\Gamma_{X_a}|_{\min} \ge \left[1 + d\left(1 - \frac{1}{\tilde{n}}\cdot\frac{r-s}{r}|X|\right)\right]\frac{r-s}{r}|X| \qquad (6)$$

Substituting $\tilde{n}$ into $nr/(r+s)$, we get:

$$|\Gamma_{X_a}|_{\min} \ge \left[1 + d\left(1 - \frac{r-s}{2r}\right)\right]\frac{r-s}{r}|X|. \qquad (7)$$

$$d \ge \frac{2rs}{r^2 - s^2} \qquad (8)$$

From ineq.(7), the minimum value of $|\Gamma_{X_a}|_{\min}$ corresponds to $|X| = n/2$ can be found when ineq. (8) is taken. So, $|\Gamma_{X_a}|_{\min} \ge |X|$. In the above two cases, the proof of $|\Gamma_X| \ge |X|$ are finished.

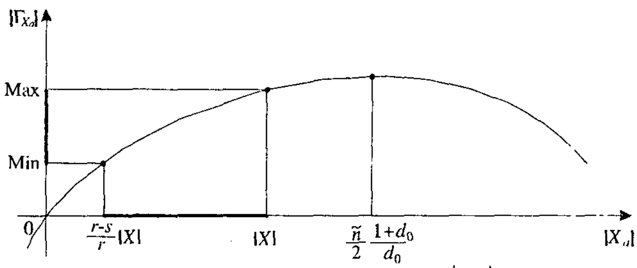

Fig.4. The Max and Min value of $|\Gamma_{X_a}|$

Lemma 1 : For all n an $(n, r/(r+s)+\varepsilon_n, k', 1/2)$ bounded concentrator can be constructed, where $0 \le \varepsilon_n = O(n^{-1/2})$.

Proof: Suppose that n' is the smallest integer larger than n. The number $n'r/(r+s)$ does not need to be an even square number, but a square number. The number $nr/(r+s)$ is also a square number. Assume $nr/(r+s) = N^2$, where N is natural number. And let

$$n = \hat{n}(r+s) \qquad (9)$$

, where $\hat{n}$ is also natural number. Then,we get from eq.(9):

$$n' - n = O(\hat{n}^{1/2}) \implies O(n^{1/2}) \qquad (10)$$

The concentrator that is satisfied with such condition can be constructed for all n, according as $(n'-n)$ inputs of *part B* are removed as shown in Fig.5. To show that $r/(r+s)+\varepsilon_n$, the ratio is followed by using eq.(10):

$$\frac{\#\text{ of output}}{\#\text{ of input}} = \frac{r}{r+s} + \varepsilon_n$$

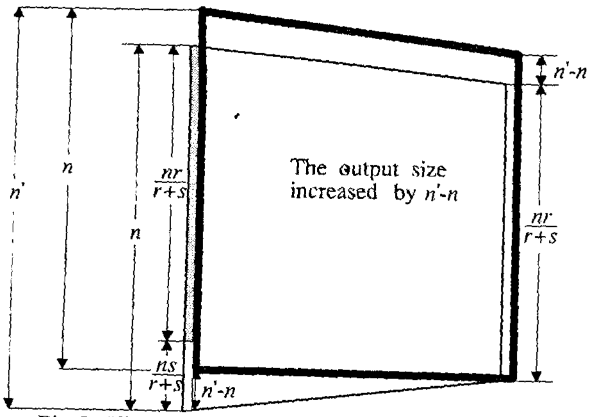

Fig.5. The construction increased by n'-n in outputs

In order to construct a superconcentrator from concentrator, following Pippenger, a network is built with n inputs and n outputs, with a direct connection from each input to a corresponding output. Moreover superconcentrating a set of inputs A to a set of outputs B where $|A|=|B|$, connect any inputs in A to any output in B that happens to be linked by the direct connection. If $|A|>n/2$, then at most $n/2$ of these inputs will fail to link using the direct connection. These are then passed through an (n,θ,k) concentrator, while on the output side a mirror image structure feeds the outputs. Between these two structures, a recursion of the entire superconcentrator structure is implemented, but with θn inputs and θn outputs. The total hardware cost $S(n)$ of this structure, in terms of the number of links, is given by:

$$S(n) = n + 2kn + S(\theta n) \tag{11}$$

or, after solving the recursion eq.(11) (ignoring the minor impact of restrictions on the number of inputs to the concentrators):

$$\frac{S(n)}{n} = \frac{2k+1}{1-\theta} \tag{12}$$

Gabber and Galil construct a obvious family of linear superconcentrators as shown in Fig.6. of density with eq.(12) for an $(n,\theta_n,k,1/2)$ bounded concentrator. For example, applying this formula to Pippenger's $(n,2/3,6,1/2)$ bounded concentrator, that proved its nonconstructive existence, yields $S(n)/n=39$. It is simple recursive construction that the density 39 drive a family of linear superconcentrators.

Theorem 2 : By an $[nr/(r+s),k,2rs/(r^2-s^2)]$ expander for every n, the density of a family of linear superconcentrators is $(2k+3)r/s+1$.

proof : It can be obviously obtained by eq.(3) and eq.(12).

$$\frac{S(n)}{n} = (2k+3)\frac{r}{s}+1$$

5. Density in Improved Concentration Coefficient

The value of r/s directly impacts the density of the resulting superconcentrator. Using the formula developed by Gabber and Galil, they found that for an expander structure with $k=5$, and $d=(2-\sqrt{3})/4$, $p=31$ and the resulting superconcentrator has density:

$$\frac{S(n)}{n} = (2k+3)p+1 = 404$$

Using the improved formula, $r/s=30$ and it can be obtained that:

$$\frac{S(n)}{n} = (2k+3)\frac{r}{s}+1 = 391$$

Gabber and Galil also developed an expander with $k=7$ and $d=(2-\sqrt{3})/2$, for which they found $p=16$, and $S(n)/n=273$. Using the improved formula we found:

$$\frac{2rs}{r^2-s^2} \le 2d \Rightarrow r/s > 14.994$$

Therefore, if we take r/s=15 since r/s is the smallest natural number such that $2d \ge 2rs/(r^2-s^2)$, the density is :

$$\frac{S(n)}{n} = (2k+3)\frac{r}{s}+1 = 256$$

This can also be compared to the methods of Gabber and Galil's. If r/s could be any real number, the value of r/s is able to be selected from below eq.

$$2d = 2rs/\left(r^2-s^2\right).$$

If $r/s=14.994$, the improved density is $255.91 \cong 256$, while they found $s(n)/n=271.77$

6. Conclusions

This paper proposes a new formula to find expansion constant, which improve the density of the concentrator and superconcentrator composed of expander. The better result is important when it projected to the most efficient well-known concentrators such as Pippenger's network. It, however, remains to be proven or disproved whether the improved bound is the best possible construction of a concentrator using an expander or not. Supposed that an expander on the hardware complexity of the Pippenger structure can be explicitly constructed, this improvement will have useful importance. The above improvement can be applied to any expander, it is not limited to the structures in [4].

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