# 멀티레이트 샘플치 시스템: 최적 디지털 재설계 기법

김도완* 박진배*, 장권규**, 주영훈**
* 연세대학교 전기전자공학과 ** 군산대학교 전자정보공학부

# Multirate Sampled-Data Control System: Optimal Digital Redesign Approach

Do Wan Kim*, Jin Bae Park*, Kwon Kyu Jang**, and Young Hoon Joo**
* Yonsei University ** Kunsan National University

**Abstract** - This paper studies a multirate sampled-data control for LTI systems by using the digital redesign (DR) method. In this note, to well tackle the problem associated with both the state matching and the stabilization, our nobel strategy is to minimize the linear quadratic cost function. The main features of the proposed method are that i) the delta-operator-based descretization method is applied to improve the state-matching performance in the fast sampling limit and/or the large input multiplicity; ii) the proposed multirate control scheme can improve the state-matching performance in the long sampling limit; iii) some sufficient conditions that guarantee the stability of the closed-loop discrete-time system and provide a guarantee cost for the cost function can be formulated in the LMIs format; and iv) an optimal sampled-data controller in the sense of minimizing the upper bound of the cost function is also given by means of an LMI optimization procedure.

**Key Words** : Digital redisign(DR), sampled-data control, linear matrix inequalities (LMIs), multirate sampling.

## 1. Introduction

Digital redesign (DR) has gained tremendously increasing attention as yet another efficient design tool of sampled-data control [1]-[6]. The term DR involves converting a well-designed analog control into an equivalent digital one maintaining the property of the original analog control system in the sense of state-matching, by which the benefits of both the analog control and the advanced digital technology can be achieved.

This paper studies a multirate sampled-data control for LTI systems by using the DR method. In this note, we attempt to minimize the linear quadratic cost function for well tackling the problem associated with both the state matching and the stabilization, The main features of the proposed method are four-fold. First, the delta-operator-based descretization method is applied to improve the state-matching performance in the fast sampling limit and/or the large input multiplicity. Second, the proposed multirate control scheme can improve the state-matching performance in the long sampling limit. Third, some sufficient conditions that guarantee the stability of the closed-loop discrete-time system and provide a guarantee cost for the cost function can be formulated in the LMIs format. Finally an optimal sampled-data controller in the sense of minimizing the upper bound of the cost function is also given by means of an LMI optimization procedure.

## 2. Preliminaries

### 2. 1. Analog Control Systems

The plant is assumed to be LTI and have a state-space representation described by

$$\frac{d}{dt}x_c(t)=Ax_c(t)+Bu_c(t) \quad (1)$$

where $x(t)$ is the state vector, and $u_c(t)$ is the digital control input.

The feedback controller is given by

$$u_c(t)=K_c x_c(t) \quad (2)$$

Then, the closed-loop system can take the form

$$\frac{d}{dt}x_c(t)=(A+BK_c)x_c(t) \quad (3)$$

Its solution at $t\in[kT, kT+T)$ is expressed by

$$x_c(kT+T)=\phi x_c(kT) \quad (4)$$

### 2. 2. Sampled-Data Control Systems

Multirate digital feedback control systems consisting of the plant and a state feedback digital controller are described by

$$\frac{d}{dt}x_d(t)=Ax_d(t)+Bu_d(t) \quad (5)$$

where $x(t)$ is the state vector, and $u_d(t)$ is the digital control input. $u_d(t)$ is interconnected by A/D converter and D/A converter in which the operation speed of D/A converter is faster than one of A/D converter. If the periods of A/D and D/A are referred to $T$ and $T'$, respectively, the control actions alternate with a shorter period $T'$ and an input multiplicity $N(=T/T')$ . And also, the digital control signals are fed into the plant with the ideal zeroth-order holds. Then, $u_d(t)$ is described by

$$u_d(t) = u_d(kT, lT') \quad (6)$$

for $t \in [kT+lT', kT+lT'+T'), l \in Z_{[0,N-1]} = \{0,1,\cdots,N-1\}$, where

$$u_d(kT, lT') = K_l x(kT, 0)$$

## 3. Main Results

### 3.1 Fast Discretization of Sampled-data System Using Delta-operator

To develop the discretized version of (1), we apply the the fast discretization technique [7] to the sampled-data system (1). The fast discretization leads to a multirate discrete-time system which can be lifted to a single-rate discrete-time system. In specific, a multirate discrete-time plant model of (1) is first derived, and then a lifted system is constructed.

Connecting the fast-sampling operator and the fast-hold operator with $[kT+lT', kT+lT'+T'), (k,l) \in Z_{\geq 0} \times Z_{[0,N-1]}$, to (1) leads the multirate discrete-time plant model. The next results presents the multirate discretized version of (1).

***Proposition 1:*** *The multirate discrete-time model of the sampled-data system (1) can be given by*

$$\delta x_d(kT, lT') = G_\delta x_d(kT, lT') + H_\delta u_d(kT, lT') \quad (8)$$

*for* $[kT+lT', kT+lT'+T'), (k,l) \in Z_{\geq 0} \times Z_{[0,N-1]}$, *where* $G_\delta = \frac{G-I}{T'}$ *and* $H_\delta = (G-I)A_i^{-1}B_i T'$.

*To transform the multirate discrete-time system (8) into the single-rate one, we invoke the discrete-time lifting. The next proposition states a discrete-time system discretized by the multirate sampling control (8).*

***Proposition 2:*** *Given a multirate discrete-time system () for* $l \in Z_{[0,N-1]}$, *a lifted sampled input*

$$\bar{u}_d(kT,0) = \begin{bmatrix} u_d(kT,0) \\ u_d(kT,T') \\ \vdots \\ u_d(kT,NT'-T') \end{bmatrix} \quad (9)$$

*leads a lifted system*

$$\delta x_d(kT,0) = \bar{G}_\delta x_d(kT,0) + \bar{H}_\delta \bar{u}_d(kT,0) \quad (10)$$

*for* $t \in [kT, kT+T)$), *where*

$$\bar{G}_\delta = \frac{G^N - I}{T},$$

$$\bar{H}_\delta = \frac{1}{T}[\, G^{N-1}H \;\; G^{N-2}H \;\cdots\; H]$$

***Corollary 1:*** *In the analog control system (3),*

- *the multirate discrete-time model can take the form*

$$\delta x_c(kT, lT') = \phi_\delta x_c(kT, lT') \quad (11)$$

*for* $t \in [kT+lT', kT+lT'+T')$, *where* $\phi_\delta = \frac{\phi - I}{T'}$.

- *the lifted system can take the form*

$$\delta x_c(kT,0) = \bar{\Phi}_\delta x_c(kT,0) \quad (12)$$

*for* $t \in [kT, kT+T)$, *where* $\bar{\Phi}_\delta = \frac{\phi^N - I}{T}$.

### 3.2 Optimal Digital Redeign

Our goal is to find $K_l$ so that the closed-loop state $x_d(t)$ matches the closed-loop state $x_c(t)$ at every sampling instant $T'$ as closely as possible, with guaranteed stability. To consider the closed-loop state $x_c(t)$ at every sampling instant $T'$, transferring (12) from $T$ to $T'$ leads

$$\delta \bar{x}_c(kT, 0) = \bar{\phi}_\delta x_c(kT,0) \quad (13)$$

where

$$\delta \bar{x}_c(kT,0) = \left[ \frac{x_c(kT,T')^T - x_c(kT,0)^T}{T'}, \cdots, \frac{x_c(kT,NT')^T - x_c(kT,0)^T}{NT'} \right]^T$$

and

$$\bar{\phi}_\delta = \begin{bmatrix} \bar{\phi}_{\delta 0} \\ \bar{\phi}_{\delta 1} \\ \vdots \\ \bar{\phi}_{\delta N-1} \end{bmatrix} = \begin{bmatrix} \frac{\phi - I}{T'} \\ \frac{\phi^2 - I}{2T'} \\ \vdots \\ \frac{\phi^N - I}{NT'} \end{bmatrix}$$

In the same manner, we can capture the state $x_d(t)$ at $T'$ by transferring from (10) to the following model:

$$\delta \bar{x}_d(kT,0) = \bar{G}_\delta x_d(kT,0) + \bar{H}_\delta \bar{u}_d(kT,0) \quad (14)$$

where

$$\delta \bar{x}_d(kT,0) = \left[ \frac{x_d(kT,T')^T - x_d(kT,0)^T}{T'}, \cdots, \frac{x_d(kT,NT')^T - x_d(kT,0)^T}{NT'} \right]^T$$

$$\bar{G}_\delta = \begin{bmatrix} \bar{G}_{\delta 0} \\ \bar{G}_{\delta 1} \\ \vdots \\ \bar{G}_{\delta N-1} \end{bmatrix} = \begin{bmatrix} \frac{G - I}{T'} \\ \frac{G^2 - I}{2T'} \\ \vdots \\ \frac{G^N - I}{NT'} \end{bmatrix}$$

and

$$\bar{H}_\delta = \begin{bmatrix} \bar{H}_{\delta 0} \\ \bar{H}_{\delta 1} \\ \vdots \\ \bar{H}_{\delta N-1} \end{bmatrix} = \begin{bmatrix} \frac{H}{T'} & 0 & \cdots\cdots & 0 \\ \frac{GH}{2T'} & \frac{H}{2T'} & 0 \;\cdots & 0 \\ \vdots & \vdots & \cdots\cdots & 0 \\ \frac{G^{N-1}H}{NT'} & \frac{G^{N-2}H}{NT'} & \cdots\cdots & \frac{H}{NT'} \end{bmatrix}$$

In sequel, our main problem for the system (14) is the problem of designing a multirate feedback control law

$$\bar{u}_d(kT,0) = \bar{K}_d x(kT,0) \quad (15)$$

where $\bar{K}_d = [K_0^T \, K_1^T \cdots K_{N-1}^T]^T$, such that i) the origin $x=0$ is a globally asymptotically stable equilibrium point of the closed-loop system

$$\delta \tilde{x}_d(kT,0) = (\bar{G}_\delta + \bar{H}_\delta \mathcal{K}_d) x_d(kT,0) \qquad (16)$$

and ii) by comparing (13) and (16), to realize $\delta \tilde{e}(kT,0) = \delta \tilde{x}_c(kT,0) - \delta \tilde{x}_d(kT,0) = 0$ under the assumption $x_c(kT,0) = x_d(kT,0)$ is numerically synthesized for $\|\delta \tilde{e}(kT,0)\|$ to be a minimizer in the induced 2-norm sense.

To well tackle the problem associated with both the state matching and the stabilization problems, our strategy is to minimize the following cost function

$$J = \sum_{k=0}^{\infty} \left( x_d^T(kT,0) Q x_d(kT,0) + \delta \tilde{e}^T(kT,0) R \delta \tilde{e}(kT,0) \right) \qquad (17)$$

where $Q \geq 0$ and $R > 0$ are given weighting matrices.

***Remark 1:*** *Minimizing $J$ corresponds in some sense to keeping $x_d(kT,0)$ and $\delta \tilde{e}(kT,0)$ close to zero. If it is more important to us that the state $x_d(kT,0)$ be small, then we should choose $Q$ large to weight it heavily in $J$, which we are trying to minimize. If it is more important that the error state $\delta \tilde{e}(kT,0)$ be small for the state matching, then we should select $R$ to be $R \gg Q$.*

From (13) and (16), $J$ in (17) for every $l \in Z_{[0,N-1]}$ can take the form

$$J = \sum_{k=0}^{\infty} x_d^T(kT,0) \left( Q + (\bar{\phi}_{\delta l} - \bar{G}_{\delta l} - \bar{H}_{\delta l} \mathcal{K}_d)^T \times R (\bar{\phi}_{\delta l} - \bar{G}_{\delta l} - \bar{H}_{\delta l} \mathcal{K}_{dl}) \right) x_d(kT,0) \qquad (18)$$

To achieve an optimal digital redesign, we consider the multirate sampled-data control problem of minimizing an upper bound of $J$ in (18) for the system (13) and (16).

***Theorem 1:*** *For the system (13) and (16), if there exist $Z$, $V$, $X = X^T > 0$, and $\mathcal{M}_d$ with appropriate dimension, such that the linear matrix inequalities (19), where $(\cdot)^T$ denotes the transposed element in symmetric position, hold then, $\mathcal{K}_d = \mathcal{M}_d Z^{-1}$ is a stabilizing periodic feedback gain with which $J$ in (18) of the closed-loop system (16) satisfies*

$$J \leq x_0^T \frac{X}{T} x_0$$

*Furthermore, if $x_0$ is a random variable satisfying*

$$\mathrm{E}\{x_0\} = 0, \qquad \mathrm{E}\{x_0 x_0^T\} = I$$

*where $\mathrm{E}\{\cdot\}$ denotes the expectation operator. an optimal feedback gain can be obtained by solving the following optimization*

$$\text{minimize}_{Z, X, \mathcal{M}_d, V} \quad \text{trace}(V)$$

*subject to linear matrix inequalities ( ) and*

$$\begin{bmatrix} -V & (\cdot)^T \\ I & -X \end{bmatrix} < 0$$

## 4. Concluding Remark

This paper studies a multirate sampled-data control for LTI systems by using the digital redesign (DR) method. This paper proposed that nobel strategy to well tackle the problem associated with both the state matching and the stabilization is to minimize the linear quadratic cost function.

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## References


[1] Y. H. Joo, L. S. Shieh, and G. Chen, ''Hybrid state-space fuzzy model-based controller with dual-rate sampling for digital control of chaotic systems,'' *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 4, pp. 394-408, 1999.

[2] W. Chang, J. B. Park, Y. H. Joo, and G. Chen, "Design of sampled-data fuzzy-model-based control systems by using intelligent digital redesign," *IEEE Trans. Circ. Syst. I*, vol. 49, no. 4, pp. 509-517, 2002.

[3] W. Chang, J. B. Park, and Y. H. Joo, ''GA-based intelligent digital redesign of fuzzy-model-based controllers,'' *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 1, pp. 35-44, 2003.

[4] W. Chang, J. B. Park, H. J. Lee, and Y. H. Joo, ''LMI approach to digital redesign of linear time-invarint systems,'' *IEE Proc., Control Theory Appl.*, vol. 149, no. 4, pp. 297-302, 2002.

[5] H. J. Lee, J. B. Park, and Y. H. Joo, ''An efficient observer-based sampled-data control: Digital redesign approach,'' *IEEE Trans. Circuits Syst. I*, vol. 50, no. 12, pp. 1595-1601, 2003.

[6] H. J. Lee, H. Kim, Y. H. Joo, W. Chang, and J. B. Park, ''A new intelligent digital redesign for T--S fuzzy systems: global approach,'' *IEEE Trans. Fuzzy Syst.,*} vol. 12, no. 2, pp. 274-284, 2004.

[7] T. Chen and B. Francis, ''Optimal Sampled-Data Control Systems,'' *Springer*, 1995.


$$\begin{bmatrix} \frac{X}{T} - Z - Z^T & (\cdot)^T & (\cdot)^T & (\cdot)^T \\ R^{\frac{1}{2}} \bar{\phi}_{\delta l} Z - R^{\frac{1}{2}} \bar{G}_{\delta l} Z - R^{\frac{1}{2}} \bar{H}_{\delta l} \mathcal{M}_d & -I & (\cdot)^T & (\cdot)^T \\ Q^{\frac{1}{2}} Z & 0 & -I & (\cdot)^T \\ \frac{Z}{T} + \bar{G}_{\delta l} Z + \bar{H}_{\delta l} \mathcal{M}_d & 0 & 0 & -\frac{X}{T} \end{bmatrix} < 0 \qquad l \in N_{[0,N-1]} \qquad (19)$$