Generalized Intuitionistic Fuzzy Matrices

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Abstract

Using the idea of generalized intuitionistic fuzzy set, we study the notion of generalized intuitionistic fuzzy matrices as a generalization of fuzzy matrices. We show that some properties of a square generalized intuitionistic fuzzy matrix such as reflexivity, transitivity and circularity are carried over to the adjoint generalized intuitionistic fuzzy matrix.

1. Introduction

In 1965, Zadeh [20] introduced the concept of fuzzy sets which formed the fundamental of fuzzy mathematics. Since then various workers have con tributed to the development of the fuzzy theory. In particular, using the idea of fuzzy sets, Kim et al. [12,13] introduced fuzzy matrices as a generalization of matrices over the two element Boolean algebra (matrices with elements having values anywhere in the closed interval [0,1]). Ragab and Emam [15] further studied some properties of the determinant and adjoint of square fuzzy matrix defined by Thomason [17] and Kim [13], respectively. Atanassov [1-4] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Using the idea of intuitionistic fuzzy sets, Im et al. [8] defined the concept of intuitionistic fuzzy matrices as a natural generalization of fuzzy matrices and in [9], they introduced and studied the determinant of square intuitionistic fuzzy matrices. Recently, Mondal and Samanta [14] introduced definitions of generalized intuitionistic fuzzy sets, as a generalization of intuitionistic fuzzy sets, generalized intuitionistic fuzzy relations and generalized intuitionistic fuzzy topology and studied some of their properties.

In this paper, using the idea of generalized intuitionistic fuzzy set", we study the notion of generalized intuitionistic fuzzy matrices as a generalization of fuzzy matrices. We show that some properties of a square generalized intuitio -nistic fuzzy matrix such as reflexivity, transitivity and circularity are carried over to the adjoint generalized intuitionistic fuzzy matrix. Finally, we prove that $A(\det A)$ is transitive for generalized intuitionistic fuzzy matrix A. It enables us to construct transitive generalized intuitionistic fuzzy matrix from a given one and it is useful for studying transitive generalized intuitionistic fuzzy matrix and generalized intuitionistic fuzzy relations [14].

2. GIF matrices

A *generalized intuitionistic fuzzy matrix* (briefly, GIF *matrix*) A is

$$\boldsymbol{A} = [(A, B)] = [(a_{ij}, b_{ij})]$$

where A and B are fuzzy matrices, and

$a_{ij} \wedge b_{ij} \leq \frac{1}{2}$ for all i, j.

Obviously, every fuzzy matrix $A = [(a_{ij})]$ is an intuitionistic fuzzy matrix of the form $[(a_{ij}, 1 - a_{ij})]$. Every intuitionistic fuzzy matrix $\boldsymbol{A} = [(a_{ij}, b_{ij})]$ is GIF matrix, since $a_{ij} + b_{ij} \leq 1$ implies $a_{ij} \wedge b_{ij} \leq \frac{1}{2}$ for all i, j.

Let $\boldsymbol{A} = [(a_{ij}, b_{ij})]$ and $\boldsymbol{c} = [(c_{ij}, d_{ij})]$ be $m \times n$ GIF matrices and $E = [(e_{ij}, f_{ij})]$ be an $n \times l$ GIF matrix. Then the matrix operations defined by

(1) $\boldsymbol{A} + \boldsymbol{C} = [(a_{ij} \vee c_{ij}, b_{ij} \vee d_{ij})]$;

(2) $\boldsymbol{AE} = [(\vee_{1 \leq k \leq n}(a_{ik} \wedge e_{kj}), \wedge_{1 \leq k \leq n}(b_{ik} \vee f_{kj}))]$;

(3) $\boldsymbol{A}^T = [(a_{ji}, b_{ji})]$;

(4) $\boldsymbol{A}^{k+1} = \boldsymbol{A}^k\boldsymbol{A}, k = 0, 1, 2, \ldots$;

(5) $\boldsymbol{A} < \boldsymbol{C}$ if $a_{ij} \leq c_{ij}$ and $b_{ij} \geq d_{ij}$ for all i, j.

Let J be an $n \times n$ fuzzy matrix that have all entries 1 and I be an $n \times n$ identity fuzzy matrix, and $\boldsymbol{I} = [(I, J - I)]$. Then by the simple calculation

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$$

Therefore, $\boldsymbol{I}$ is the *identity GIF matrix*.

Let P be an $n \times n$ permutation fuzzy matrix and $\boldsymbol{P} = [(P, J - P)]$. Then by the simple calculation

$$\boldsymbol{PP}^T = \boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I} .$$

Therefore, $\boldsymbol{P}$ is a *permutation GIF matrix*.

Theorem 2.1 *Let $\boldsymbol{A} = [(a_{ij}, b_{ij})]$, $\boldsymbol{C} = [(c_{ij}, d_{ij})]$ and $\boldsymbol{E} = [(e_{ij}, f_{ij})]$ be $n \times n$ GIF matrices.* If $\boldsymbol{A} < \boldsymbol{C}$, then $\boldsymbol{AE} < \boldsymbol{CE}$.

Theorem 2.2 *Let $\boldsymbol{A} = [(A, B)]$ be a GIF matrix and $\boldsymbol{P}$ be a permutation GIF matrix. Then $\boldsymbol{PA}$ is a row changed matrix of $\boldsymbol{A}$ and $\boldsymbol{AP}$ is a column changed matrix of $\boldsymbol{A}$.*

Definition 2.3 The *determinant* $\det\boldsymbol{A}$ of an $n \times n$ GIF matrix $\boldsymbol{A} = [(a_{ij}, b_{ij})]$ is defined as follows:

$$\det\boldsymbol{A} = (\vee_{\sigma \in S_n} a_{1\sigma(1)} \wedge \cdots \wedge a_{n\sigma(n)}, \wedge_{\sigma \in S_n} b_{1\sigma(1)} \vee \cdots \vee b_{n\sigma(n)}),$$

where S_n denotes the symmetric group of all permutations of the indices $\{1, 2, \cdots, n\}$.

Theorem 2.4 *If a GIF matrix $\boldsymbol{C}$ is obtained from an $n \times n$ GIF matrix $\boldsymbol{A} = [(A, B)]$ by multiplying $\boldsymbol{A}$ by $k \in (0, 1]$, then $\det\boldsymbol{C} = k(\det\boldsymbol{A})$.*

Theorem 2.5 *Let $\boldsymbol{A} = [(A, B)]$ be an $n \times n$ GIF matrix. Then*

$$\det(\boldsymbol{I}_{ij}\boldsymbol{A}) = \det\boldsymbol{A} = \det(\boldsymbol{AI}_{ij}) ,$$

where $\boldsymbol{I}_{ij}$ is a permutation GIF matrix which is obtained from the identity GIF matrix by interchanging row i and row j.

Since any permutation GIF matrix is the product of $\boldsymbol{I}_{ij}$'s for some i and j we have the following:

Corollary 2.6 *Let $\boldsymbol{A} = [(A, B)]$ be an $n \times n$ GIF matrix. Then*

$$\det(\boldsymbol{PAQ}) = \det\boldsymbol{A},$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are any permutation GIF matrices.

From Corollary 2.7, we know that

$$\det(\boldsymbol{PA}) = (\det\boldsymbol{P})(\det\boldsymbol{A})$$

where $\boldsymbol{P}$ is a permutation GIF matrix and $\boldsymbol{A}$ is any GIF matrix. However, in general we have the following:

Theorem 2.7 $\det(\boldsymbol{AB}) > (\det\boldsymbol{A})(\det\boldsymbol{B})$ for any GIF matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

3. The adjoint of square GIF matrices

Definition 3.1 The *adjoint matrix* of an $n \times n$ GIF matrix $\boldsymbol{A} = [(A, B)]$, denoted by adj $\boldsymbol{A}$, is defined as follows:

$$\mathrm{adj}\boldsymbol{A} = [(c_{ij}, d_{ij})] = [(\det A_{ji}, \det B_{ji})],$$

where $(\det A_{ji}, \det B_{ji})$ is the determinant of the $(n-1) \times (n-1)$ GIF matrix formed by deleting row j and column i from A and B, respectively, in the each operations.

We know that $\det A_{ji}$ can be obtained from $\det A$ by replacing a_{ji} by 1 and all other row-j factors a_{jk}, $k \neq i$, by 0 and det B_{ji} can be obtained from $\det B$ by replacing b_{ji} by 0 and all other row-j factors b_{jk}, $k \neq i$, by 1. We also can write the elements c_{ij} and d_{ij} of $\mathrm{adj}\boldsymbol{A}$ as follows:

$$c_{IJ} = (\vee_{\sigma \in S_{n_j n_i}} \wedge_{t \in n_j} a_{t\sigma(t)}),$$

$$d_{ij} = (\wedge_{\sigma \in S_{n_j n_i}} \vee_{t \in n_j} d_{t\sigma(t)}),$$

where $n_j = 1, 2, \cdots, n-j$ and $S_{n_j n_i}$ is the set all permutations of set n_j over the set n_i.

Theorem 3.2 *Let* $\boldsymbol{A} = [(A, B)]$ *and* $\boldsymbol{C} = [(C, D)]$ *be* $n \times n$ *GIF matrices. Then*

(1) $\boldsymbol{A} < \boldsymbol{C}$ implies $adj\boldsymbol{A} < \mathrm{adj}\ \boldsymbol{C}$;

(2) $\mathrm{adj}\boldsymbol{A} + \mathrm{adj}\boldsymbol{C} < \mathrm{adj}(\boldsymbol{A} + \boldsymbol{C})$;

(3) $adj(\boldsymbol{A}^T) = (\mathrm{adj}\mathbf{A})^{\mathbf{T}}$.

Theorem 3.3 *Let* $\boldsymbol{A} = [(A, B)]$ *be an* $n \times n$ *GIF matrix. Then*

(1) $\boldsymbol{A}(\mathrm{adj}\boldsymbol{A}) > (\det\boldsymbol{A}$,

(2) $(\mathrm{adj}\boldsymbol{A})\boldsymbol{A} > (\det\boldsymbol{A})\boldsymbol{I}_n$.

Theorem 3.4 *Let* $\boldsymbol{A} = [(A, B)]$ *be a GIF matrix with a* 0*'s row in* A *and a* 1*'s row in* B *and let* O *be a zero matrix. Then* $(\mathrm{adj}\boldsymbol{A})\boldsymbol{A} = [(O, J)]$.

Theorem 3.5 *Let* $\boldsymbol{A} = [(A, B)]$ *be an* $n \times n$ *GIF matrix. Then* $\det\boldsymbol{A} = \det(\mathrm{adj}\boldsymbol{A})$.

Definition 3.6 A $m \times n$ GIF matrix $\boldsymbol{A} = [(A, B)]$ is said to be *constant* if $a_{ik} = a_{jk}$ and $b_{ik} = b_{jk}$ for all i, j, k, that is, its rows are equal to each other.

Theorem 3.7 *Let* $\boldsymbol{A} = [(A, B)]$ *be an* $n \times n$ *constant GIF matrix. Then*

(1) $(\mathrm{adj}\boldsymbol{A})^T$ *is constant*;

(2) $\boldsymbol{A}(\mathrm{adj}\boldsymbol{A})$ *is constant*;

(3) $\det A$ *is the minimum element in* A *and* $\det B$ *is the maximum element in* B.

Definition 3.8 Let $\boldsymbol{A} = [(A, B)]$ be an GIF matrix. Then $\boldsymbol{A}$ is called

(1) *symmetric* if $\boldsymbol{A} = \boldsymbol{A}^T$;

(2) *reflexive* if $\boldsymbol{A} > \boldsymbol{I}_n$;

(3) *transitive* if $\boldsymbol{A}^2 < \boldsymbol{A}$;

(4) *idempotent* if $\boldsymbol{A}^2 = \boldsymbol{A}$;

(5) *circular* if $(\boldsymbol{A}^2)^T < \boldsymbol{A}$.

Lemma 3.9 *Let* $\boldsymbol{A}$ *be an* $n \times n$ *reflexive GIF matrix. Then* $\mathrm{adj}\boldsymbol{A} = \boldsymbol{A}^c$ *where* $\boldsymbol{A}^c$ *is idempotent and* $c \leq n-1$.

Theorem 3.10 *Let* $\boldsymbol{A} = [(A, B)]$ *be an* $n \times n$ *reflexive GIF matrix. Then*

(1) $\mathrm{adj}\boldsymbol{A}$ *is reflexive*;

(2) $\mathrm{adj}\boldsymbol{A} > \boldsymbol{A}$;

(3) $\mathrm{adj}\boldsymbol{A}^2 = (\mathrm{adj}\boldsymbol{A})^2 = \mathrm{adj}\boldsymbol{A}$;

(4) *if* $\boldsymbol{A}$ *is idempotent, then* $\mathrm{adj}\boldsymbol{A} = \boldsymbol{A}$;

(5) $\mathrm{adj}(\mathrm{adj}\boldsymbol{A}) = \mathrm{adj}\boldsymbol{A}$;

(6) $\boldsymbol{A}(\mathrm{adj}\boldsymbol{A}) = \mathrm{adj}\boldsymbol{A} = (\mathrm{adj}\boldsymbol{A})\boldsymbol{A}$.

Theorem 3.11 *For an* $n \times n$ *GIF* $\boldsymbol{A} = [(A, B)]$ *we have the following:*

(1) *If* $\boldsymbol{A}$ *is symmetric, then* $\mathrm{adj}\boldsymbol{A}$ *is*

symmetric.

(2) *If* $\mathbf{A}$ *is transitive, then* $\mathrm{adj}\mathbf{A}$ *is transitive.*

(3) *If* $\mathbf{A}$ *is circular, then* $\mathrm{adj}\mathbf{A}$ *is circular.*

Theorem 3.12 *If* $\mathbf{A} = [(A, B)]$ *is an* $n \times n$ *GIF matrix, then GIF matrix* $\mathbf{A}(\mathrm{adj}\mathbf{A})$ *is transitive.*

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