부호가 있는 구간치 쇼케이 적분

Signed interval-valued Choquet integrals

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요약

본 논문에서, 우리는 부호가 있는 구간치 쇼케이적분을 정의하고 부호가 있는 수간치 쇼케이 적분이 이산과 단조성이 없는 경우를 모델화할 수 있는가를 보인다. 더욱이 일시적인 선택, 재화 가격과 복지평가 등의 응용에 관해서도 언급하고자한다.

Abstract

In this paper, we define signed interval-valued Choquet integrals and shows the signed interval-valued Choquet integrals can model violations of separability and monotonicity. Furthermore, we discuss some applications to intertemporal preference, asset pricing, and welfare evauations.

Keywords : fuzzy measures, signed Choquet integrals, interval-valued functions.

1. Introduction.

Aumann([2]) first defined the concept of integrals of set-valued functions(simply, Aumann's intgerals) with respect to a classical measure and the ingale theory of set-valued functions and many other fields. In [4,5], we defined closed set-valued Choquet integrals with respect to a fuzzy measure and some basic properties of them. For real-valued Choquet integrals, Sugeno at el have studied applications of Choquet integrals with respect to a fuzzy measure in [12, 13, 14, 15]. In order to study applications of closed set-valued Choquet integrals, we have been trying to modify our papers [4, 5] and obtained some properties of closed set-valued Choquet integrals, for example, convergence theorems for closed set-valued Choquet integrals under Hausdorff's convergence. Mostly recently, Zhang, Guo and Liu([21]) restudied our paper [5] and proved convergence theorem for closed set-valued Choquet integrals under Kuratowski's convergence. Thus we also studied basic properties of *interval-valued* Choquet integrals and discussed their characterizations.

In this paper, we consider the signed interval number-valued Choquet integrals which is a generalization of the signed Choquet integrals in the papers [16, 17] and discuss basic properties of them. Furthermore, we investigate how the signed interval-valued Choquet integral can be applied to interval-valued intertemporal preferences, modeling inetractions between periods that can be so strong that monotonicity is violated.

2. Definitions and preliminaries.

Let $\{1, \cdots, n\}$ be a finite set of time points. Interval-valued profiles n-tuples $(X_1, \cdots, X_n)$ also denoted by X. We note that interval-valued profiles mean uncertain profiles and we can calculate Choquet integrals of interval-valued profiles.

Throughout the paper, R is the set of real numbers and

$$I(R) = \{[a, b] \mid a, b \in R \text{ and } a \leq b\}.$$

Then a element in $I(R)$ is called an interval number.

On the interval number set, we define; for each pair $[a,b],[c,d]\in I(R)$ and $k\in R$,

$$[a,b]+[c,d]=[a+c,\ b+d]$$

$$[a,b]\cdot[c,d]=[a\cdot c\wedge a\cdot d\wedge b\cdot c\wedge b\cdot d,\ a\cdot c\vee a\cdot d\vee b\cdot c\vee b\cdot d]$$

$$k[a,b]=\begin{cases}[ka,kb], & k\ge 0\\ [kb,ka], & k<0\end{cases}$$

$[a,b]\le[c,d]$ if and *only* if $a\le c$ and $b\le d$

$$max_{1\le i\le n}[a_i,b_i]=[\max_{1\le i\le n}a_i,\ \max_{1\le i\le n}b_i]$$

We note that $(I(R),d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_H(A,B)=\max\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\}$$

for all $A,B\in I(R)$ (see [8-11]).

We remark that an interval-valued profile means uncertain profile. They also can be interpreted as interval-valued functions from $\{1,\cdots,n\}$ to $I(R)$, describing uncertain consumption or uncertain income at each time point. Waegenaere([15-20]) investigated the result to real-valued profiles. For decision under uncertainty, time points are reinterpreted as states of nature and interval-valued profiles as acts, for welfare theory time points are persons and interval-valued profiles are welfare allocations.

In this paper, our main idea is that acts and welfare allocations have uncertainty. Using our main idea, we show how the signed interval-valued Choquet integral can be applied to intertemporal interval-valued preference, for examples, the interval-valued equlibrium and the interval-valued equilibrium price. Interval-valued profiles $X=(X_1,\cdots,X_n)$ and $Y=(Y_1,\cdots,Y_n)$ are comonotonic, denoted by $X\sim Y$ if there are no time points i,j such that $x_i^*>x_i^*$ $(x_{i*}>x_{i*})$ and $y_i^*<y_j^*$ $(y_{i*}<y_{j*})$, where

$$x_i^*=\max X_i,\ x_{i*}=\min X_i,\ y_i^*=\max Y_i \text{ and } y_{i*}=\min Y_i.$$

We note that $X\sim Y$ if and only if $x^*\sim y^*$ and $x_*\sim y_*$. A subset of $[I(R)]^n$ is comonotonic if every pair of interval-valued profiles in the subset is comonotonic. A fuzzy measure on a measurable space $(\Omega,\mathfrak{I})$ is an extended real-valued function $\mu:\mathfrak{I}\to[0,\infty]$ satisfying

(i) $\mu(\emptyset)=0$,

(ii) $\mu(A)\le\mu(B)$, whenever $A,B\in\mathfrak{I}$, $A\subset B$.

A fuzzy measure μ is said to be lower semi-continuous if for every increasing sequence $\{A_n\}$ of measurable sets, we have $\mu(\cup_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$. A fuzzy measure μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1)<\infty$, we have $\mu(\cap_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$. If μ is both lower semi-continuous and upper semi-continuous, it is said to be continuous(see [10]).

Definition 2.1 ([12-17]) (1) The signed Choquet integral of a measurable function x with respect to a fuzzy measure μ is defined by

$$(C)\int xd\mu=\int_0^{\infty}\mu(\{w\in\Omega\,|x(w)\ge r\})dr+\int_{-\infty}^{0}[\mu(\{w\in\Omega\,|x(w)\ge r\})-\mu(\Omega)]dr.$$

(2) If Ω is a finite set $\Omega=N=\{1,\cdots,n\}$, the signed Choquet integral of x can be written as follows:

$$(C)\int xd\mu=\int_0^{\infty}\mu(\{j\in N\,|x(j)\ge r\})dr+\int_{-\infty}^{0}[\mu(\{j\in N\,|x(j)\ge r\})-\mu(N)]dr.$$

Now we define the signed interval-valued Choquet integral as in the following.

Definition 2.2 The signed Choquet integral of a interval-valued function X with respect to a fuzzy measure μ is defined by

$$(C)\int Xd\mu=\{(C)\int xd\mu\ |\ x\in S(X)\}$$

where $S(F)$ is the family of $\mu-a.e.$ measurable selections of F.

We consider Kuratowski's convergence in the Definitions 3.6 and 3.6' ([10]). Then we have the following theorem whose proof is similar to the proof of Theorem 3.10([10]).

Theorem 2.3 Let μ be a continuous fuzzy measure and X a measurable and Choquet integrably bounded set-valued function.

(1) If X is closed set-valued, then $(C)\int Xd\mu$ is clsoed.

(2) If X is convex set-valued, then $(C)\int Xd\mu$ is

convex.

(3) If X is interval-valued, i.e. $X(w)=[x_*(w), x^*(w)]$, $w \in \Omega$, then

$$(C)\int X d\mu = [(C)\int x_* d\mu, (C)\int x^* d\mu].$$

We also discuss comonotonic additivity of a signed interval-valued Choquet integral.

Theorem 2.4 Let μ be a continuous fuzzy measure and X a measurable and Choquet integrably bounded set-valued function. Then the signed interval-valued Choquet integral of X satisfies comonotonic additivity.

In this case the following theorem for calculating the signed interval-valued Choquet integrals is useful. It follows from the preceding equation by integration by parts and Theorem 2.3.

Theorem 2.4 If we take a permutation ρ on N that is compatible with X, i.e. $X_{\rho(1)} \geq \cdots \geq X_{\rho(n)}$ and define

$$\pi_{\rho(j)} = \mu(\{\rho(1), \cdots, \rho(j)\}) - \mu(\{\rho(1), \cdots, \rho(j-1)\})$$

for all j, then we have

$$(C)\int X d\mu = \sum_{j=1}^{n} \pi_{\rho(j)} X_{\rho(j)},$$

where $X_{\rho(j)} = [x_{*\rho(j)}, x^*_{\rho(j)}]$, $x_{*\rho(j)} = \min X_{\rho(j)}$, and $x^*_{\rho(j)} = \max X_{\rho(j)}$.

We note that the numbers π_j are called decision weights and that if $\mu(N)=1$, then the signed interval-valued Choquet integral of $X_j=[\alpha,\beta]$ with $\alpha \leq \beta (\alpha, \beta \in R)$ is $[\alpha,\beta]$.

3. The application of interval-valued Choquet integrals.

In this section, we consider a two period asset market model with dealers charging bid-ask spreads, show that interval-valued equilibrium exists, and give a characterization of interval-valued equilibrium prices of the assets. We note that interval-valued equilibrium mean uncertain equilibrium prices, that is, we can consider uncertain protfolios(resp., uncertain consumption bundles) instead of certain protfolios(resp., certain consumption bundles). We can see that the interval-valued price of an asset can be represented by the interval-valued Choquet integral.

There are J nominal assets, indexed by $j \in J = \{1, \cdots, J\}$. The assets can be traded in the first period, and yield payoff in the second period. There S possible states of the world at the second period, indexed by $s \in S = \{1, \cdots, S\}$. For simplicity of notation, we assume that there are no spot markets, i.e. there is only one good at each state of the world, and assets yield payoff in quantities of this good. An interval-valued consumption bundle is a vector

$$X = (X_0, X_1, \cdots, X_S)^t \in [I(R)]^{S+1},$$

consisting of X_0 units of the good in the second period if state s occurs, for $s \in S$. The payoff of asset $j \in J$ is denoted by a vector $A_j \in R^S$. The matrix of asset payoffs is denoted by $A \in R^{S \times J}$.

There are I agents, indexed by $i \in I = \{1, \cdots, I\}$ with interval-valued utility functions

$$U^i: [I(R)]^{S+1} \rightarrow I(R).$$

They have initial interval-valued endowments $W^i = (W^i_0, W^i_1, \cdots, W^i_S)^t$ and maximize interval-valued utility by trading asset interval-valued portfolios $Z=(Z_1, \cdots, Z_J)^t \in [i(R)]^J$. Again for simplicity of notation, we assume that there is only one dealer. The presence of a dealer is formalized by the fact that for each asset A_j, $j \in J$, there is a buying interval-valued price $Q(A_j)$ and a selling interval-valued price $-Q(-A_j)$. Typically, one will have that $Q(A_j) > -Q(-A_j)$, i.e. the dealer can make an interval-valued profit equal to interval-valued bit-ask spread $\Gamma_j = Q(A_j) + Q(-A_j)$ by buying the asset from an agent for the interval-valued price $-Q(-A_j)$ and selling it to an agent for the interval-valued price $Q(A_j)$.

Furthermore, when an interval-valued portfolio consisting of more than one asset is traded, the dealer takes into account that hedging effects can reduce the risk of the interval-valued portfolio. Consequently, he might allow an interval-valued price discount in this case. More precisely, for an interval-valued portfolio

$Z \in [I(R)]^J$, in general one will have that

$$Q(AZ)+Q(-AZ) \leq \sum_{j=1}^{J} | Z_j | \, (Q(A_j)+Q(-A_j))$$

i.e. the interval-valued spread on an interval-valued portfolio is less than or equal to the sum of the individual interval-valued spreads.

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