The Rate of Change of an Energy Functional for Axially Moving Continua

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Abstract: In this paper, with the utilization of a three-dimensional version of Leibniz's rule, the procedure of deriving the time rate of change of an energy functional for axially moving continua is investigated. It will be shown that the method in [14], which describes the way of getting the time rate of change of an energy functional in Eulerian description, and subsequent results in [10, 11] are not complete. The key point is that the time derivatives at boundaries in the Eulerian description of axially moving continua should take into account the velocity of the moving material itself. A noble way of deriving the time rate of change of the energy functional is proposed. The correctness of the proposed method has been confirmed by other approaches. Two examples, one-dimensional axially moving string and beam equations, are provided for the purpose of demonstration. The results following the procedure proposed and the results in [14] are compared.

Keywords: Axially moving continua, Lyapunov method, Leibniz's rule, Eulerian and Lagrangian descriptions, energy functional.

1. INTRODUCTION

Axially moving continua (continuous materials) can be found in various engineering areas: high speed magnetic tapes, band saws, power transmission chains and belts, steel strips in the galvanizing line, paper sheets under processing, etc. Especially, the dynamics analysis and control of axially moving continua have received a growing attention due to the entrance of a new era of flexible robotic manipulators and flexible space structures. However, the utilization of axially moving continua is limited because of the unwanted vibrations of the system in many applications, and in particular in high-speed precision systems.

Active vibration control is then an important solution to reduce vibration and improve performance in many of the axial transport systems. Vibration control schemes on axially moving strings include references [3, 6, 7, 8, 10, 13, 16, 17, etc.], and those on axially moving beams include references [2, 5, 9, 11, etc.]. Most of the control laws in above references have been derived by using the Lyapunov method.

As well known, the most important issue in the Lyapunov method is the selection of a suitable energy functional and then the construction of an effective control law through the time derivative of the energy functional. Thus, it is essential that the time derivative of the energy functional considered should be exactly performed with a proper mathematical manner.

Renshaw et al. [14] have suggested a differentiation method in Eulerian description for the energy functionals of prototypical axially moving string and beam models, and have concluded that a conserved Eulerian functional is the Jacobi integral of the system and qualifies as a Lyapunov functional when it is positive definite. In the literature, their conclusions have been accepted as an established theory for calculating the time derivative of energy functionals of the axially moving systems (see, [10, 11]).

But, unfortunately, the method presented in [14] and subsequent results in [10, 11] are not complete because the time derivative of the energy functional in Eulerian description should have taken into account the velocity of the moving material.

In this paper, the reason why the result in [14] is not complete is explained in detail in Section 2. In Section 3, a three-dimensional version of the Leibniz's rule is employed as a novel method to calculate the time derivative of an energy functional for axially moving continua. From the Leibniz's rule, it is seen that the time derivative of the energy functional in Eulerian description should involve the velocity of the moving material. Also, two energy functionals for one-dimensional axially moving string and beam models and their time derivatives are derived and then compared with those obtained with the method in [14].

2. PROBLEM FORMULATION

Fig. 1 shows the schematic of an axially moving string, which will be used as a representative example for developing a theory and comparison. The two support rolls at both boundaries are assumed fixed, i.e., fixed in the sense that there is no vertical movement but it allows the string to move in the horizontal direction.

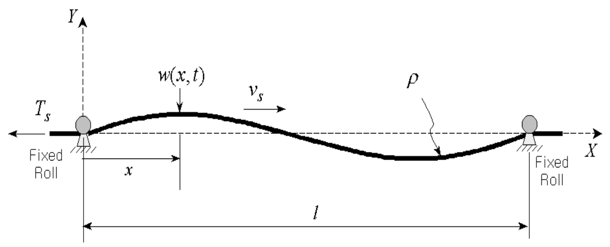

Fig 1. Schematic of an axially moving string with fixed boundaries.

Let t be the time, x be the spatial coordinate along the longitude of motion, v_s be the axial speed of the string, $w(x,t)$ be the transversal displacement of the string at spatial coordinate x and time t, and l be the length of the string from the left to the right supports. Also, let ρ be the mass per unit length and T_s be the tension applied to the string.

Because the string travels with a constant speed, v_s, the total derivative operator (material derivative) with respect to time should be defined as

$$\dot{(\cdot)} \overset{\Delta}{=} \frac{d(\cdot)}{dt} = \frac{\partial(\cdot)}{\partial t} + \frac{dx}{dt}\frac{\partial(\cdot)}{\partial x} = (\cdot)_t + v_s(\cdot)_x , \quad (1)$$

where $(\cdot)_t = \partial(\cdot)/\partial t$ and $(\cdot)_x = \partial(\cdot)/\partial x$ denote the partial derivatives: $(\cdot)_t$ defines the rate of change at a fixed place, i.e., the local change, and $(\cdot)_x$ gives the change due to the advancement from x to $x+dx$, i.e., the convective change.

The mechanical energy of the string between $x=0$ and $x=l$ is given by

$$V_{string} = \frac{1}{2}\int_0^l \rho(v_s w_x + w_t)^2 \, dx + \frac{1}{2}\int_0^l T_s w_x^2 \, dx , \quad (2)$$

where $w_x \overset{\Delta}{=} w_x(x,t)$ and $w_t \overset{\Delta}{=} w_t(x,t)$ have been abbreviated. It is noted that (2) comprises the kinetic energy in the transverse direction plus the potential energy due to tension. The kinetic energy due to the longitudinal movement v_s has been excluded because it does not affect the final form of the control law that will be derived. By using the Hamilton's principle, the governing equation and boundary conditions of the axially moving string are derived as follows:

$$\rho w_{tt} + 2\rho v_s w_{xt} + \rho v_s^2 w_{xx} - T_s w_{xx} = 0 , \quad 0 < x < l ,$$
$$w(0,t) = w(l,t) = 0 . \quad (3)$$

In the process of designing a controller using the Lyapunov method, it is essential to analyze and handle the time derivative of a Lyapunov function candidate of the system considered, i.e., the mechanical energy. Renshaw et al. [14] have presented that to get the time derivative of an energy functional for axially moving continua, only an Eulerian functional is qualified as a Lyapunov functional candidate under the assumption that it is positive definite, but not a Lagrangian functional.

The Eulerian description of the mechanical energy of the span $[0, l\,]$ in Fig. 1 is

$$V_{Eul} = \frac{1}{2}\int_0^l \left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right] dx . \quad (4)$$

Alternatively, the Lagrangian description of the mechanical energy of the set of particles between $x = v_s t$ and $v_s t + l$ is

$$V_{Lag} = \frac{1}{2}\int_{v_s t}^{v_s t + l} \left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right] dx , \quad (5)$$

where $V_{Lag} = V_{Eul}$ at $t=0$ .But, note that dV_{Eul}/dt and dV_{Lag}/dt are distinct. The main assertions in [14] are summarized as follows:

For dV_{Eul}/dt, the following result was claimed.

$$\frac{dV_{Eul}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right] dx$$
$$= \frac{1}{2}(T_s v_s - \rho v_s^3) w_x^2 \Big|_0^l , \quad (6)$$

where the first equality has been derived using one-dimensional Leibniz rule of the following form

$$\frac{d}{dt}\int_{A(t)}^{B(t)} f(t,s) ds = \int_{A(t)}^{B(t)} \frac{\partial}{\partial t} f(t,s) ds + f(t, B(t)) B'(t)$$
$$- f(t, A(t)) A'(t) , \quad (7)$$

and (3) has been used in deriving the second equality. The first equality in (6) can be also derived from the following equation

$$\frac{dV_{Eul}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right] dx$$
$$+ \frac{1}{2}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]\frac{dx}{dt}\Big|_0^l , \quad (8)$$

if $dx/dt = 0$ is treated zero at $x = 0, l$.

For dV_{Lag}/dt, on the other hand, the limits of integration are time dependent. The result given in Renshow et al. (1988) is

$$\frac{dV_{Lag}}{dt} = \frac{1}{2}\int_{v_s t}^{v_s t + l} \frac{\partial}{\partial t}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right] dx$$
$$+ \frac{1}{2}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]\frac{dx}{dt}\Big|_{v_s t}^{v_s t + l}$$
$$= \frac{1}{2}(T_s v_s - \rho v_s^3) w_x^2 \Big|_{v_s t}^{v_s t + l}$$
$$+ \frac{1}{2}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]\frac{dx}{dt}\Big|_{v_s t}^{v_s t + l}$$
$$= T_s v_s w_x^2 \Big|_{v_s t}^{v_s t + l} , \quad (9)$$

where $w_t = 0$ at both boundaries due to the fixed boundary conditions.

Renshow et al. [14] have observed that (9) is valid only at $t=0$ because (3) applies only when the material particles associated with V_{Lag} comprise the string span $[0, l\,]$, and hence (9) is a valid energy functional only at $t = 0$. Consequently, their conclusions were: A positive definite Lagrangian functional, even though it is a material derivative, can not be used as a Lyapunov functional, because its time derivative is not valid for more than an instant. Instead, the Eulerian functional of (6) should be used as a Lyapunov functional even though it is not a material derivative.

Remark 1: The observation in [14] that the Lagrangian functional can not be a Lyapunov function is correct. But, the way of evaluating the Eulerian functional (6) is not correct. In the literature a numerous results [6, 7, 8, 9, 17, etc] have relied on the material derivatives, which are all believed correct.

Remark 2: The assertion in [14] misses a critical fact in evaluating (6), i.e., dx/dt must not be considered zero at $x = 0, l$ even though 0 and l are constants. It is believed that such a conclusion was due to the misunderstanding that dx/dt denotes the velocity of the surface of the span part $[0, l\,]$ in the Eulerian description. However, in fact dx/dt in (8) denotes the velocity of the material being transmitted, not the velocity of the support point. Therefore, the correct evaluation of (6) should be given in the following form:

$$\frac{dV_{Eul}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]dx$$
$$+\frac{1}{2}v_s\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]_0^l$$
$$= T_s v_s w_x^2\Big|_0^l, \quad (10)$$

which will be explained in detail in the next section.

3. RATE OF CHANGE: THE CORRECT METHOD

For the time derivative of an energy functional of axially moving continua, a three-dimensional version of Leibniz's rule is first derived. For this, we introduce a time-varying volume $U(t)$ of finite magnitude with a time-varying surface $S(t)$ that encloses $U(t)$. Note that "time-varying" means "moving and/or deforming" while "time-invariant" means "fixed," i.e., "neither moving nor deforming."

The axially moving continua have to be analyzed in view of Eulerian description since our attention is focused on what happens on a specific region of space as time passes. Further, the volume and the surface that are occupied by a specific region of the axially moving span should be considered as time-varying since the axially moving continua vibrate even in the case that the both boundaries are fixed.

At first, it is necessary to deal with a function $\varphi(t)$ defined by an integral of the form

$$\varphi(t) = \int_{U(t)} \psi(\vec{x},t)\,dU\,, \quad (11)$$

where $\vec{x}$ is the position vector relative to a chosen origin and the quantity ψ represents the fluxes of mass, linear momentum, angular momentum, internal energy, and kinetic energy, etc. In (11), the volume integral is a triple integral. Note that the position vector $\vec{x}$ is a time independent variable in Eulerian description while the vector $\vec{x}$ is a time dependent variable in Lagrangian description (see [15, pp. 8-11; 4, pp. 5-8]).

Since a material point in Eulerian description is described at place $\vec{x}$ and time t, $\vec{x}$ is called the field coordinate and $d\vec{x}/dt = \vec{u}$ is called *the material velocity given in field coordinates* (see [15, pp. 8-11; 4, pp. 5-8]). Note that using the material derivative as defined in (1) the velocity $\vec{u}$ can be also obtained as follows:

$$\frac{d\vec{x}}{dt} = \frac{\partial\vec{x}}{\partial t} + \vec{u}\cdot\nabla\vec{x} = \vec{u}\,, \quad (12)$$

where $\partial\vec{x}/\partial t = 0$ since $\vec{x}$ and t are independent variables, and $\vec{u}\cdot\nabla\vec{x} = \vec{u}$ since the gradient of $\vec{x}$, or $\nabla\vec{x}$, is the unit dyadic (see [4, pp. 6; 12]). On the other hand, *the material velocity in Lagrangian description* is given as $\partial\vec{x}/\partial t = \vec{u}$, where $\vec{x}$ is a time dependent variable which is attached to the material point that moves (see [15, pp. 9; 4, pp. 8]).

Now, an expression for the time derivative $d\varphi/dt$, i.e., a three-dimensional version of Leibniz rule is derived yielding a more direct interpretation than that of [4] for our need.

If U in (11) is time-invariant, then the differentiation with respect to t under the integral sign, $d\psi/dt$, can be justified for all t on a time interval when ψ and the resultant integrand $\partial\psi/\partial t$ are continuous for $a \le t \le b$ and the domain of integration.

More generally, when $U(t)$ is time-varying, $\varphi(t)$ can be considered as a function of t directly and also indirectly, through the intermediate variable $S(t)$ which denotes the limits of $U(t)$ in the volume integral. Note that the region of $S(t)$ is occupied by material points at time t and each material point on the surface is also described as the field coordinate $\vec{x}$ at time t in the Eulerian description. Hence φ can be written as $\varphi = \varphi(\vec{x},t)$. It then follows by employing the total differential that

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \nabla\varphi\cdot\vec{u}\,, \quad (13)$$

where ∇-operator provides the gradient of φ. If the derivatives on the right side in (13) are continuous, then $\partial\varphi/\partial t$ is to be calculated by treating U as a time-invariant or fixed region, and hence by merely differentiating with respect to t under the integral sign when ψ and $\partial\psi/\partial t$ are continuous. To evaluate the other partial derivative of φ in (13), i.e., $\nabla\varphi\cdot\vec{u}$, Gauss' theorem holds:

$$\nabla\varphi = \int_U \nabla\psi\,dU = \oint_S \psi\,\hat{n}\,dS\,, \quad (14)$$

where $\hat{n}$ is the outwardly positive unit normal vector to $S(t)$, and the surface integral is a double integral while the volume integral is a triple integral. Note that the time-varying surface region, $S(t)$, is occupied by the material with the velocity $\vec{u}$ on the surface during the motion.

Consequently, the Leibniz's rule in three dimensions is given by

$$\frac{d}{dt}\int_{U(t)}\psi\,dU = \int_{U(t)}\frac{\partial}{\partial t}\psi\,dU + \oint_{S(t)}\psi\vec{u}\cdot\hat{n}\,dS\,, \quad (15)$$

where the volume integral on the right side in (15) represents the change in ψ that occurs within $U(t)$, while the surface integral accounts for the transport of ψ across the time-varying surface $S(t)$.

Note that the surface integral in (15), $\oint_{S(t)}\psi\vec{u}\cdot\hat{n}\,dS$, denotes the net flux of ψ that crosses $S(t)$ since the amount of ψ that crosses a differential area dS of $S(t)$ per unit time is $\psi\vec{u}\cdot\hat{n}\,dS$.

The most important point in the application of (15) is: In the Leibniz's rule of (15), $\vec{u}$ denotes *the material velocity given in field coordinates*, not the velocity of time-varying surface $S(t)$ bounding the time-varying volume $U(t)$. From this reason, even though the boundary conditions (supports) in Fig. 1 are fixed, dx/dt in (8) should not be treated as zero.

The same conclusion can be drawn from other view as follows: Suppose that $\vec{u}$ in (15) denote only the velocity of $S(t)$, then it means that the following equation, known as Reynold's transport theorem, is no longer valid:

$$\frac{d}{dt}\int_{U(t)}\psi\, dU = \int_U \frac{\partial}{\partial t}\psi\, dU + \oint_S \psi\vec{u}\cdot\hat{n}\, dS, \qquad (16)$$

where the terms on the right side in (16) have the same physical interpretation as those on the right side of (15). From (16), it is seen that the time-varying region $U(t)$ on the right side is replaced with a fixed region U (and a fixed surface S) which coincide with it at time t. Note that the system with U and S on the right side in (16) seems like an open system since mass may cross the fixed surface S with the material velocity $\vec{u}$. Otherwise, i.e., if $\vec{u}$ in (16) denotes the velocity of S, then (16) should be given as

$$\frac{d}{dt}\int_{U(t)}\psi\, dU = \int_U \frac{\partial}{\partial t}\psi\, dU, \qquad (17)$$

due to $\vec{u}=0$ under the condition of a fixed surface.

Actually, (16) has been derived from the following equations by using the Leibniz's rule given by (15)

$$\frac{d}{dt}\int_M \psi\, dm = \int_M \frac{d}{dt}\psi\, dm, \qquad (18)$$

where m denotes the mass of the bounded piece of material and is the sum of the mass elements dm over the set M of the material point of the closed material volume $U(t)$, i.e., $m=\int_M dm$ (see [15, pp. 30-35]). Note that the integral in (18) is a single integral and that M in (18) is constant since the same piece of material is always considered by the closed surface $S(t)$. Thus, if following the theory presented in [14], then (18) should be given as

$$\frac{d}{dt}\int_M \psi\, dm = \int_M \frac{\partial}{\partial t}\psi\, dm, \qquad (19)$$

because the domain of integration M is constant, i.e., the limits of integration are time independent. From (19), it is impossible to derive the result in (16). It just means that the Reynold's transport theorem given by (16) cannot be applied any more.

As investigated above, it is evident that $\vec{u}$ in (15) should not be considered as the velocity of the surface $S(t)$ and also this consideration is unsuitable even in view of the physical concept. That is, $\vec{u}$ in (15) should denote the material velocity.

Finally, the time derivative of one-dimensional axially moving continua can be obtained through the three-dimensional version of Leibniz's rule given by (15) as follows: During the vibrations, the shape of the volume changes due to the moving and deforming and successively takes up new regions in space. Hence, $U(t)$ denote the region which is occupied by the axially moving span part at time t. However, the surface $S(t)$ is given as two points in the one-dimensional case, i.e., limits of the integration.

Example 1: For the one-dimensional axially moving string introduced in (3), the material velocity $\vec{u}$ is $\vec{v}_s$. Note that in this system the material points at $x=0,l$ still have the material velocity $\vec{v}_s$ despite of fixed boundary conditions, and $\vec{v}_s\cdot\hat{n}$ at the boundaries are given as $-v_s$ at $x=0$ and $+v_s$ at $x=l$ because $\hat{n}$ is the outwardly positive unit normal vector to $S(t)$.

Thus, the time derivative of (2) by using (15) is given as

$$\dot{V}_{string} = \frac{dV_{string}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]dx + \frac{1}{2}v_s\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]_0^l$$
$$= T_s v_s w_x^2\Big|_0^l. \qquad (20)$$

Note that (20) is the same as (10) and entirely different from the results in (6) and (9), which were presented in [14].

Example 2: (15) is now applied to the time derivative of an energy functional of one-dimensional axially moving beam with axial tension T_s, flexural rigidity EI, and traveling speed v_s. The governing equation of the beam is

$$\rho w_{tt} + 2\rho v_s w_{xt} + \rho v_s^2 w_{xx} - T_s w_{xx} + EI w_{xxxx} = 0,$$
$$0 < x < l. \qquad (21)$$

The mechanical energy is defined by

$$V_{beam} = \frac{1}{2}\int_0^l \rho(v_s w_x + w_t)^2\, dx + \frac{1}{2}\int_0^l T_s w_x^2\, dx + \frac{1}{2}\int_0^l EI w_{xx}^2\, dx. \qquad (22)$$

The time derivative using (15) is given as

$$\dot{V}_{beam} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2 + EI w_{xx}^2\right]dx + \frac{1}{2}v_s\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2 + EI w_{xx}^2\right]_0^l. \qquad (23)$$

More specific forms of $\dot{V}_{beam}$ with various boundary conditions are derived as follows:

Case 1: Fixed boundary conditions, i.e.,

$$w(0,t) = w(l,t) = w_x(0,t) = w_x(l,t) = 0:$$

$$\dot{V}_{beam} = EI v_s w_{xx}^2\Big|_0^l. \qquad (24)$$

Case 2: Simply supported conditions, i.e.,

$$w(0,t) = w(l,t) = w_{xx}(0,t) = w_{xx}(l,t) = 0:$$

$$\dot{V}_{beam} = (T_s v_s w_x^2 - EI v_s w_x w_{xxx})\Big|_0^l. \qquad (25)$$

Remark 3: From (20) and (23), the material derivative defined in (1) can be directly utilized in the derivation of (20) and (23) as follows:

$$\dot{V}(t) = \frac{d}{dt}\int_0^l \tilde{V}(x,t)dx = \int_0^l \frac{d}{dt}\tilde{V}(x,t)dx$$
$$= \int_0^l\left[\frac{\partial}{\partial t}\tilde{V}(x,t) + v_s\frac{\partial}{\partial x}\tilde{V}(x,t)\right]dx$$
$$= \int_0^l \frac{\partial}{\partial t}\tilde{V}(x,t)dx + v_s\tilde{V}(x,t)\Big|_0^l, \qquad (26)$$

where $\tilde{V} = \left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]/2$ and $\tilde{V} = \left[\rho(v_s w_x + w_t)^2 + T_s w_x^2 + EI w_{xx}^2\right]/2$ in the case of string system (3) and beam system (21), respectively. However, the result in (26) should be understood from the viewpoint that it is really obtained through (15) because of the vibrations of the continua. It is also noted that the result in (26) is entirely different from that of the Lagrangian description with the material derivative in (9).

Remark 4: The result in (20) also coincides with the result derived by using the Lyapunov stability theorem [1, pp. 46-47] as follows:

Let $V(w_x, w_t, t)$ a Lyapunov function candidate of the axially moving string in (3) such that

$$V(w_x, w_t, t) = \frac{1}{2}\int_0^l \left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]dx\,, \qquad (27)$$

where $V(w_x, w_t, t) = V_{string}$ in (2). Then, following the Lyapunov stability theorem the time derivative of $V(w_x, w_t, t)$ is obtained as

$$\dot{V}(w_x, w_t, t) = \frac{d}{dt}V(w_x, w_t, t)$$

$$= \frac{\partial}{\partial t}V + \frac{\partial}{\partial w_x}V\frac{d}{dt}w_x(x,t) + \frac{\partial}{\partial w_x}V\frac{d}{dt}w_x(x,t) \qquad (28)$$

$$= \frac{1}{2}\int_0^l \frac{\partial}{\partial w_x}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]\left(\frac{\partial w_x}{\partial t} + \frac{dx}{dt}\frac{\partial w_x}{\partial x}\right)dx$$
$$+ \frac{1}{2}\int_0^l \frac{\partial}{\partial w_t}\left[\rho(v_s w_x + w_t)^2 + T_s w_x^2\right]\left(\frac{\partial w_t}{\partial t} + \frac{dx}{dt}\frac{\partial w_t}{\partial x}\right)dx \qquad (29)$$

$$= \int_0^l \left[\rho(v_s w_x + w_t)v_s + T_s w_x\right](w_{xt} + v_s w_{xx})dx$$
$$+ \int_0^l \left[\rho(v_s w_x + w_t)\right](w_{tt} + v_s w_{xt})dx$$

$$= T_s v_s w_x^2\Big|_0^l\,. \qquad (30)$$

The result in (30) obtained by the Lyapunov stability theorem is the same as that in (20) not that in (6). Note that the derivation method employed in (28)-(29) is being accepted in general in the literature for distributed parameter systems (see [1, pp. 45-63]).

Remark 5: Further, note that (29) is in the form of (8). If dx/dt in (29) is zero but not v_s as given in (1), then it means that V_{string} in (2) can be analyzed as

$$V_{string} = \frac{1}{2}\int_0^l \left[\rho v_s^2 w_x^2 + 2\rho v_s w_x w_t + \rho w_t^2 + T_s w_x^2\right]dx$$
$$= \frac{1}{2}\int_0^l \left[\rho v_s w_x\left(\frac{dx}{dt}\frac{\partial w}{\partial x}\right) + 2\rho w_t\left(\frac{dx}{dt}\frac{\partial w}{\partial x}\right) + \rho w_t^2 + T_s w_x^2\right]dx\,, \qquad (31)$$

where w_x and w_t are assumed constant. Then (31) should be given as follows:

$$V_{string} = \frac{1}{2}\int_0^l \left[\rho w_t^2 + T_s w_x^2\right]dx + \frac{1}{2}(\rho v_s w_x)\, w\frac{dx}{dt}\Big|_0^l$$
$$+ (\rho w_t)\, w\frac{dx}{dt}\Big|_0^l$$
$$= \frac{1}{2}\int_0^l \left[\rho w_t^2 + T_s w_x^2\right]dx\,, \qquad (32)$$

where $dx/dt = 0$. The result in (32) is the same as the mechanical energy of a tensioned string without axial moving. However, it is impossible in the physical concept to explain that the mechanical energy of an axially moving string with constant w_x and w_t is the same as that of the tensioned string without axial moving. Thus, this gives another evidence that the method presented in [14] is not correct.

4. CONCLUSIONS

In this paper, the rate of change of an energy functional for axially moving continua, which involve a time-varying volume and a time-varying surface, has been investigated. The axially moving continua have been analyzed in view of Eulerian description since our attention is focused on what happens at a specific region of space as time passes. Even though an axially moving system has fixed boundary conditions, the volume and surface of the system cannot be considered as time-invariant (or fixed) because of the vibrations of the material part. Thus, the three-dimensional version of the Leibniz's rule has been established and correctly applied to get the correct result on the time derivative of the energy functional. The material velocity given in field coordinates has been taken into account, but not the velocity of the time-varying surface bounding the time-varying volume.

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