Isotropic Configurations of Omnidirectional Mobile Robots with Three Caster Wheels

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Abstract: In this paper, we identify the isotropic configurations of an omnidirectional mobile robot with three caster wheels, depending on the selection of actuated joints. First, we obtain the kinematic model of a caster wheeled omnidirectional mobile robot(COMR) without matrix inversion. For a given task velocity, the instantaneous motion of each wheel is decomposed into two orthogonal instantaneous motions of the steering and the rotating joints. Second, with the characteristic length introduced, we derive the isotropy conditions of a COMR having n (≥ 3) actuated joints, which are imposed on two Jacobian matrices, $\mathbf{A} \in R^{n\times 3}$ and $\mathbf{B} \in R^{6\times 6}$. Under the condition of $\mathbf{B} \propto \mathbf{I}_6$, three caster wheels should have identical structure with the length of the steering link equal to the radius of the wheel. Third, depending on the selection of actuated joints, we derive the conditions for $\mathbf{A}^t \mathbf{A} \propto \mathbf{I}_3$ and identify the isotropic configurations of a COMR. All possible actuation sets with different number of actuated joints and different combination of rotating and steering joints are considered.

Keywords: Caster wheeled omnidirectional mobile robot, kinematic model, isotropy analysis

1. INTRODUCTION

The era of personal robots expects robots to serve humans for house care, health care, entertainment & education, public welfare, etc. Typically, personal robots are requested to have omnidirectional mobility to navigate at human walking speed in daily life environment that is restricted in space and cluttered with stationary and moving obstacles.

Several omnidirectional wheel mechanisms have been proposed, including universal wheels, Swedish wheels, orthogonal wheels, and ball wheels. Recently, caster wheels were employed to develop an omnidirectional mobile robot at Stanford University [1], which was commercialized by Nomadic Technologies as XR4000. Since caster wheels do without small peripheral rollers or support structure, a caster wheeled omnidirectional mobile robot(COMR) can maintain good performance as payload or ground condition changes.

There have been several works on a COMR. [2] showed that at least four joints of two caster wheels should be actuated to avoid the singularity. [3] proposed the transfer method for the kinematic modeling of a COMR, which requires matrix inversion. [4] examined the global isotropy of a COMR for the optimal design parameters of the mechanism. On the other hand, for an omindirectional mobile robot with Swedish wheels, [5] performed the kinematic modeling and the isotropy analysis. In [5], the kinematic model are obtained without matrix inversion, however, the isotropy conditions derived are incomplete and need elaboration.

The purpose of this paper is to analyze the isotropy of a COMR. This paper is organized as follows: First, the kinematic model is obtained without matrix inversion. Second, the isotropy conditions on two Jacobian matrices are derived. Third, the isotropic configurations are identified depending on the selection of actuated joints.

2. KINEMATIC MODEL

Consider a COMR with three caster wheels attached to a regular triangular platform moving on the xy-plane, as shown in Fig. 1.

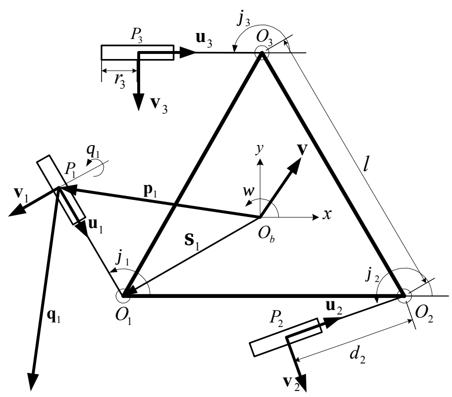

Fig. 1 A caster wheeled omnidirectional mobile robot.

Let l be the side length of the platform with the center denoted by O_b, and the vertices denoted by O_i, $i=1,2,3$. For the i^{th} caster wheel with the center denoted by P_i, $i=1,2,3$, we define the following. Let d_i and r_i be the length of the steering link and the radius of the wheel, respectively. Let φ_i and θ_i and be the steering and the rotating angles of the caster wheel, respectively. Let $\mathbf{u}_i$ and $\mathbf{v}_i$ be two orthogonal unit vectors along the steering link and the wheel axis, respectively, such that

$$\mathbf{u}_i=\begin{bmatrix} -\cos\varphi_i \\ -\sin\varphi_i \end{bmatrix}, \quad \mathbf{v}_i=\begin{bmatrix} -\sin\varphi_i \\ \cos\varphi_i \end{bmatrix} \tag{1}$$

Let $\mathbf{s}_i$ be the vector from O_b to O_i, such that

$$\mathbf{s}_1=\frac{l}{\sqrt{3}}\begin{bmatrix}-\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{bmatrix},\ \mathbf{s}_2=\frac{l}{\sqrt{3}}\begin{bmatrix}\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{bmatrix},\ \mathbf{s}_3=\frac{l}{\sqrt{3}}\begin{bmatrix}0\\1\end{bmatrix} \quad (2)$$

Let $\mathbf{p}_i$ be the vector from O_b to P_i, and $\mathbf{q}_i$ be the rotation of $\mathbf{p}_i$ by 90^o counterclockwise.

Let $\mathbf{v}$ and ω be the linear and the angular velocities at O_b of the platform, respectively. For the i^{th} caster wheel, $i=1,2,3$, the linear velocity at the point of contact with the ground can be expressed by

$$\mathbf{v} + \omega\,\mathbf{q}_i = r_i\,\dot{\theta}_i\,\mathbf{u}_i + d_i\,\dot{\varphi}_i\,\mathbf{v}_i,\quad i=1,2,3 \quad (3)$$

Premultiplying (3) by $\mathbf{u}_i^t$ and $\mathbf{v}_i^t$, respectively, we have

$$\mathbf{u}_i^t\,\mathbf{v} + \mathbf{u}_i^t\,\mathbf{q}_i\,\omega = r_i\,\dot{\theta}_i,\quad i=1,2,3 \quad (4)$$

$$\mathbf{v}_i^t\,\mathbf{v} + \mathbf{v}_i^t\,\mathbf{q}_i\,\omega = d_i\,\dot{\varphi}_i,\quad i=1,2,3 \quad (5)$$

Assuming that both steering and rotating joints of three caster wheels are actuated, the kinematics of a COMR can be written by

$$\mathbf{A}\,\dot{\mathbf{x}} = \mathbf{B}\,\dot{\boldsymbol{\Theta}} \quad (6)$$

where $\dot{\mathbf{x}} = [\ \mathbf{v}\ \omega\ ]^t \in \mathbf{R}^{3\times 1}$ is the task velocity vector, and $\dot{\boldsymbol{\Theta}} = [\dot{\theta}_1\ \dot{\varphi}_1 \cdots \dot{\theta}_3\ \dot{\varphi}_3]^t \in \mathbf{R}^{6\times 1}$ is the joint velocity vector, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{u}_1^t & \mathbf{u}_1^t\,\mathbf{q}_1 \\ \mathbf{v}_1^t & \mathbf{v}_1^t\,\mathbf{q}_1 \\ \mathbf{u}_2^t & \mathbf{u}_2^t\,\mathbf{q}_2 \\ \mathbf{v}_2^t & \mathbf{v}_2^t\,\mathbf{q}_2 \\ \mathbf{u}_3^t & \mathbf{u}_3^t\,\mathbf{q}_3 \\ \mathbf{v}_3^t & \mathbf{v}_3^t\,\mathbf{q}_3 \end{bmatrix} \in \mathbf{R}^{6\times 3} \quad (7)$$

$$\mathbf{B} = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_3 \end{bmatrix} \in \mathbf{R}^{6\times 6} \quad (8)$$

are the Jacobian matrices.

It is worthwhile to mention that our kinematic modeling of a COMR does not involve matrix inversion, unlike the transfer method proposed in [3]. For a given task velocity, the instantaneous motion of each wheel is decomposed into two orthogonal components: the instantaneous motion of the steering joint and the instantaneous motion of the rotating joint. The resulting kinematic model of a COMR allows us to perform the geometric and intuitive analysis on the isotropy of the mechanism.

3. ISOTROPY CONDITIONS

Based on (6), the isotropy condition of a COMR can be stated as

$$\mathbf{A}^t\,\mathbf{A} \propto \mathbf{I}_3 \quad (9)$$

$$\mathbf{B} \propto \mathbf{I}_6 \quad (10)$$

where $\mathbf{I}$ is the identity matrix.

3.1 Isotropy condition on B

From (8) and (10), the isotropy condition on $\mathbf{B}$ is obtained by

$$r_1 = r_2 = r_3 = d_1 = d_2 = d_3 = d > 0 \quad (11)$$

(11) tells that three caster wheels should have identical structure with the length of the steering link equal to the radius of the wheel.

3.2 Isotropy condition on A

To resolve the inconsistency in physical unit of the first two columns and the third column of $\mathbf{A}$, we introduce the characteristic length, L. From now on, it is assumed that n ($3\le n\le 6$) joints of a COMR are actuated. With $\dot{\mathbf{x}} = [\ \mathbf{v}\ L\omega\ ]^t$, $\mathbf{A}$ can be redefined as

$$\mathbf{A} = \begin{bmatrix} \mathbf{g}_1^t & \frac{1}{L}\,\mathbf{g}_1^t\,\mathbf{h}_1 \\ \vdots & \vdots \\ \mathbf{g}_n^t & \frac{1}{L}\,\mathbf{g}_n^t\,\mathbf{h}_n \end{bmatrix} \in \mathbf{R}^{n\times 3} \quad (12)$$

where $\mathbf{g}_k$ and $\mathbf{h}_k$, $k=1,\cdots,n$, correspond to $\mathbf{u}_i$ or $\mathbf{v}_i$, and $\mathbf{q}_i$, $i=1,2,3$, respectively. From (9) and (12), we have

$$\mathbf{A}^t\,\mathbf{A} = \begin{bmatrix} \sum_1^n \mathbf{g}_k\,\mathbf{g}_k^t & \frac{1}{L}\sum_1^n (\mathbf{g}_k^t\,\mathbf{h}_k)\,\mathbf{g}_k \\ \frac{1}{L}\sum_1^n (\mathbf{g}_k^t\,\mathbf{h}_k)\,\mathbf{g}_k^t & \frac{1}{L^2}\sum_1^n (\mathbf{g}_k^t\,\mathbf{h}_k)^2 \end{bmatrix} = \sigma\,\mathbf{I}_3 \quad (13)$$

Since

$$\begin{aligned}\sigma &= \frac{1}{3}\,\mathrm{tr}(\mathbf{A}^t\,\mathbf{A}) \\ &= \frac{1}{3}\left[\mathrm{tr}\Big(\sum_1^n \mathbf{g}_k\,\mathbf{g}_k^t\Big) + \frac{1}{L^2}\sum_1^n (\mathbf{g}_k^t\,\mathbf{h}_k)^2\right] \quad (14) \\ &= \frac{1}{3}[\,n + \sigma\,]\end{aligned}$$

it follows that

$$\sigma = \frac{n}{2} \quad (15)$$

Plugging (15) into (13), the following three isotropy conditions on $\mathbf{A}$ can be obtained:

$$\sum_1^n \mathbf{g}_k\,\mathbf{g}_k^t = \frac{n}{2}\,\mathbf{I}_2 \quad (16)$$

$$\sum_1^n (\mathbf{g}_k^t\,\mathbf{h}_k)\,\mathbf{g}_k = \mathbf{0} \quad (17)$$

$$\frac{1}{L^2}\sum_1^n (\mathbf{g}_k^t\,\mathbf{h}_k)^2 = \frac{n}{2} \quad (18)$$

where $\mathbf{0}$ is the zero vector. Since $\mathbf{g}_k$, $k=1,\cdots,n$, is unit vector, (16) gives two scalar constraints, while (17) and (18) give two and one scalar constraints, respectively.

Seen from Fig. 1, we have

$$\mathbf{u}_i = \mathbf{R}\,\mathbf{v}_i,\quad \mathbf{v}_i = \mathbf{R}^t\,\mathbf{u}_i,\quad i=1,2,3 \quad (19)$$

$$\mathbf{q}_i = \mathbf{R}\,\mathbf{p}_i,\quad \mathbf{p}_i = \mathbf{R}^t\,\mathbf{q}_i,\quad i=1,2,3 \quad (20)$$

$$\mathbf{p}_i = \mathbf{s}_i - d\,\mathbf{u}_i,\quad i=1,2,3 \quad (21)$$

where $\mathbf{R}=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ is the rotation matrix by 90^o counterclockwise and $\mathbf{R}^{-1}=\mathbf{R}^t$. Using (19)-(21), the expressions of $\mathbf{g}_k^t\,\mathbf{h}_k$, $k=1,\cdots,n$, can be simplified as follows: In the case of $\mathbf{g}_k = \mathbf{u}_i$ and $\mathbf{h}_k = \mathbf{q}_i$, $i=1,2,3$,

$$\mathbf{g}_k^t\,\mathbf{h}_k = \mathbf{u}_i^t\,\mathbf{q}_i = \mathbf{v}_i^t\,\mathbf{p}_i = \mathbf{v}_i^t\,\mathbf{s}_i \quad (22)$$

In the case of $\mathbf{g}_k = \mathbf{v}_i$ and $\mathbf{h}_k = \mathbf{q}_i$, $i=1,2,3$,

$$\mathbf{g}_k^t\ \mathbf{h}_k = \mathbf{v}_i^t\ \mathbf{q}_i = -\ \mathbf{u}_i^t\ \mathbf{p}_i = -\ \mathbf{u}_i^t\ \mathbf{s}_i + d \qquad (23)$$

4. ISOTROPIC CONFIGURATIONS

4.1 $\boldsymbol{\Theta}=[\theta_1\ \theta_2\ \theta_3]^t$

Consider the case where three rotating joints of three caster wheels are actuated, for which $n=3$, $\sigma=1.5$, and $[\ \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ ]=[\ \mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ ]$. Under the first isotropy condition of (16), we have

$$\sum_{k=1}^{3} \mathbf{u}_k\ \mathbf{u}_k^t = 1.5\ \mathbf{I}_2 \qquad (24)$$

which is

$$\begin{aligned} c_1^2 + c_2^2 + c_3^2 &= 1.5 \\ c_1 s_1 + c_2 s_2 + c_3 s_3 &= 0.0 \end{aligned} \qquad (25)$$

where $c_k=\cos(\varphi_k)$ and $s_k=\sin(\varphi_k)$, $k=1,2,3$. Since the same amount of rotation applied to $\mathbf{u}_k$, $k=1,2,3$, does not affect (24), φ_1 can be set arbitrarily without loss of generality.

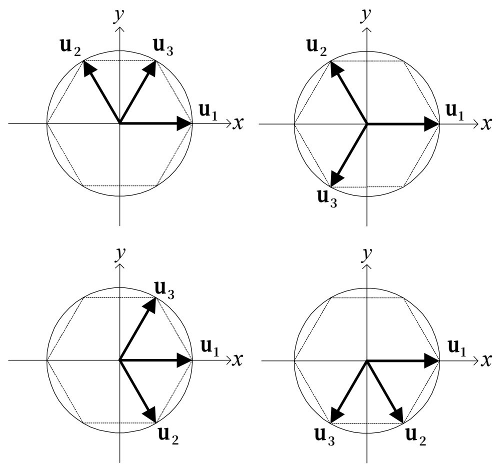

Fig. 2 Possible distributions of $\{\ \mathbf{u}_k, k=1,2,3\ \}$.

With $\varphi_1=\pi$, Fig. 2 shows one group of four distributions of $\{\ \mathbf{u}_k, k=1,2,3\ \}$ on the unit circle, which satisfy (25). It should be noted that three unit vectors are chosen to be independent out of six unit vectors directed toward the vertices of a regular 6-polygon. Four distributions shown in Fig. 2 are common in that

$$\varphi_2 = \varphi_1 + \frac{2}{3}\pi,\ \varphi_1 - \frac{\pi}{3} \qquad (26)$$

$$\varphi_3 = \varphi_1 + \frac{\pi}{3},\ \varphi_1 - \frac{2}{3}\pi \qquad (27)$$

Geometrically, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ lie on three sides of a regular triangle in counterclockwise order. On the other hand, the other group of four distributions satisfying (25) can be obtained by alternating $\mathbf{u}_2$ and $\mathbf{u}_3$ from Fig. 2. Seen from (19), the distributions of $\{\ \mathbf{v}_k,\ k=1,2,3\ \}$ can be obtained by rotating the distributions of $\{\ \mathbf{u}_k, k=1,2,3\ \}$ by 90^o clockwise. Note that all eight distributions satisfying (25) can be characterized as

$$\mathbf{v}_1 \pm \mathbf{v}_2 \pm \mathbf{v}_3 = \mathbf{0} \qquad (28)$$

Under the second isotropy condition of (17), we have

$$\sum_1^3 (\ \mathbf{u}_k^t\ \mathbf{q}_k\ )\ \mathbf{u}_k = \mathbf{0} \qquad (29)$$

Using (19) and (22), (29) is equivalent to

$$\sum_1^3 (\ \mathbf{v}_k^t\ \mathbf{s}_k\ )\ \mathbf{v}_k = \mathbf{0} \qquad (30)$$

where $\sum_{k=1}^{3} \mathbf{u}_k = \mathbf{R}\sum_{k=1}^{3} \mathbf{v}_k$ is used. In (30), $\mathbf{v}_k^t\ \mathbf{s}_k$, $k=1,2,3$, is the projection of $\mathbf{s}_k$ onto $\mathbf{v}_k$, given by

$$\begin{aligned} \mathbf{v}_1^t\ \mathbf{s}_1 &= \frac{l}{\sqrt{3}}\cos(\varphi_1 - \frac{2}{3}\pi) = \alpha_1 \\ \mathbf{v}_2^t\ \mathbf{s}_2 &= \frac{l}{\sqrt{3}}\cos(\varphi_2 + \frac{2}{3}\pi) = \alpha_2 \\ \mathbf{v}_3^t\ \mathbf{s}_3 &= \frac{l}{\sqrt{3}}\cos(\varphi_3) = \alpha_3 \end{aligned} \qquad (31)$$

using (1) and (2).

Table 1. Relationships among α_1, α_2, and α_3.

φ_2	φ_3	Relationship
$\varphi_1+\frac{2}{3}\pi$	$\varphi_1+\frac{\pi}{3}$	$\alpha_1=\alpha_2=-\alpha_3$
	$\varphi_1-\frac{2}{3}\pi$	$\alpha_1=\alpha_2=\alpha_3$
$\varphi_1-\frac{\pi}{3}$	$\varphi_1+\frac{\pi}{3}$	$\alpha_1=-\alpha_2=-\alpha_3$
	$\varphi_1-\frac{2}{3}\pi$	$\alpha_1=-\alpha_2=\alpha_3$

For one group of four distributions shown in Fig. 2, Table 1 lists the relationships among α_1, α_2, and α_3. For each of four distribution, it holds that

$$|\alpha_1| = |\alpha_2| = |\alpha_3| = \alpha \qquad (32)$$

$$\alpha_1\ \mathbf{v}_1 + \alpha_2\ \mathbf{v}_2 + \alpha_3\ \mathbf{v}_3 = \mathbf{0} \qquad (33)$$

Geometrically, $\alpha_k\ \mathbf{v}_k$, $k=1,2,3$, is directed toward the vertex of a regular triangle circumscribed by a circle of radius α. On the other hand, for the other group of four distributions, it is impossible to satisfy (32) and (33) and the isotropy of $\mathbf{A}$ cannot be achieved. For instance, in the case of $\varphi_2=\varphi_1-\frac{2}{3}\pi$ and $\varphi_3=\varphi_1-\frac{\pi}{3}$, we have $\alpha_1=\frac{l}{\sqrt{3}}\cos(\varphi_1-\frac{2}{3}\pi)$, $\alpha_2=\frac{l}{\sqrt{3}}\cos(\varphi_1)$, and $\alpha_3=\frac{l}{\sqrt{3}}\cos(\varphi_1-\frac{\pi}{3})$, so that $\sum_1^3 \alpha_k\ \mathbf{v}_k \neq \mathbf{0}$. Fig. 3 illustrates the isotropic configuration of a COMR, where three steering links are symmetric with respect to the center of the platform to form a regular triangle inscribed by a circle of radius α.

Under the third isotropy condition of (18), we have

$$\frac{1}{L^2}\sum_1^3 (\ \mathbf{u}_k^t\ \mathbf{q}_k\ )^2 = 1.5 \qquad (34)$$

or

$$\frac{1}{L^2}(\ \alpha_1^2 + \alpha_2^2 + \alpha_3^2) = 1.5 \qquad (35)$$

using (22) and (31). With (32) being held, from (35), the characteristic length of an isotropic COMR is obtained by

$$L = \sqrt{2}\,a \tag{36}$$

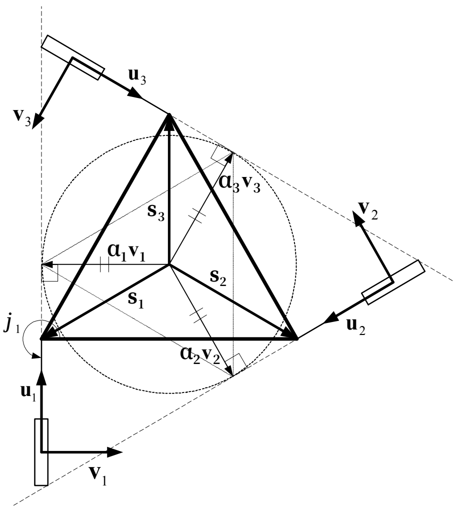

Fig. 3 Isotropic configuration for $\boldsymbol{\Theta}=[\theta_1\ \theta_2\ \theta_3]^t$.

Similar analysis to the above can be made for the case of $\boldsymbol{\Theta}=[\varphi_1\ \varphi_2\ \varphi_3]^t$ where three steering joints of three caster wheels are actuated.

4.2 $\boldsymbol{\Theta}=[\varphi_1\ \theta_2\ \theta_3]^t$

Consider the case where one steering and two rotating joints of three caster wheels are actuated, for which $n=3$, $\sigma=1.5$, and $[\ \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ ]=[\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{u}_3\ ]$. First, under the condition of (16), we have

$$\mathbf{v}_1\ \mathbf{v}_1^{\,t} + \mathbf{u}_2\ \mathbf{u}_2^{\,t} + \mathbf{u}_3\ \mathbf{u}_3^{\,t} = 1.5\ \mathbf{I}_2 \tag{37}$$

which yields eight possible distributions of $\{\ \mathbf{u}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\ \}$, characterized by

$$\mathbf{u}_1 \pm \mathbf{v}_2 \pm \mathbf{v}_3 = \mathbf{0} \tag{38}$$

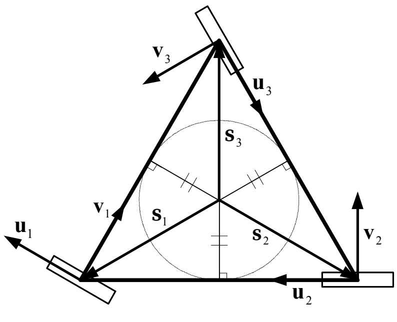

Fig. 4 Isotropic configuration for $\boldsymbol{\Theta}=[\varphi_1\ \theta_2\ \theta_3]^t$.

Next, under the condition of (17), we have

$$(\ \mathbf{v}_1^{\,t}\ \mathbf{q}_1\ )\ \mathbf{v}_1 + (\ \mathbf{u}_2^{\,t}\ \mathbf{q}_2\ )\ \mathbf{u}_2 + (\ \mathbf{u}_3^{\,t}\ \mathbf{q}_3\ )\ \mathbf{u}_3 = \mathbf{0} \tag{39}$$

which is equivalent to

$$(\ \mathbf{u}_1^{\,t}\ \mathbf{s}_1 - d\ )\ \mathbf{u}_1 + (\ \mathbf{v}_2^{\,t}\ \mathbf{s}_2\ )\ \mathbf{v}_2 + (\ \mathbf{v}_3^{\,t}\ \mathbf{s}_3\ )\ \mathbf{v}_3 = \mathbf{0} \tag{40}$$

using (22) and (23). With (38) being held, (40) requires that

$$|\ \mathbf{u}_1^{\,t}\ \mathbf{s}_1 - d\ | = |\ \mathbf{v}_2^{\,t}\ \mathbf{s}_2\ | = |\ \mathbf{v}_3^{\,t}\ \mathbf{s}_3\ | \tag{41}$$

which cannot be satisfied unless

$$d = 0 \tag{42}$$

(42) tells that the isotropy of $\mathbf{A}$ can be achieved only when caster wheels reduce to conventional wheels without steering link, as illustrated in Fig. 4.

Similar analysis to the above can be made for the case of $\boldsymbol{\Theta}=[\varphi_1\ \varphi_2\ \theta_3]^t$ where two steering and one rotating joints of three caster wheels are actuated.

4.3 $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2]^t$

Consider the case where both rotating and steering joints of one caster wheel and the rotating joint of another caster wheel are actuated, for which $n=3$, $\sigma=1.5$, and $[\ \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ ]=[\ \mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ ]$. First, under the condition of (16), we have

$$\mathbf{u}_1\ \mathbf{u}_1^{\,t} + \mathbf{v}_1\ \mathbf{v}_1^{\,t} + \mathbf{u}_2\ \mathbf{u}_2^{\,t} = 1.5\ \mathbf{I}_2 \tag{43}$$

Since $\mathbf{u}_1\ \mathbf{u}_1^{\,t} + \mathbf{v}_1\ \mathbf{v}_1^{\,t} = \mathbf{I}_2$, (43) reduces to

$$c_2^{\,2} = 0.5, \quad c_2 s_2 = 0.0 \tag{44}$$

for which there does not exist φ_2. This tells that the isotropy of $\mathbf{A}$ cannot be achieved at all.

Similar analysis to the above can be made for the case of $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \varphi_2]^t$ where both rotating and steering joints of one caster wheel and the steering joint of another caster wheel are actuated.

4.4 $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \varphi_2]^t$

Consider the case where both rotating and steering joints of two caster wheels are actuated, for which $n=4$, $\sigma=2$, and $[\ \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\ ]=[\ \mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{v}_2\ ]$. First, under the condition of (16), we have

$$\sum_{1}^{2} (\ \mathbf{u}_k\ \mathbf{u}_k^{\,t} + \mathbf{v}_k\ \mathbf{v}_k^{\,t}\ ) = 2\ \mathbf{I}_2 \tag{45}$$

which always holds.

Next, under the condition of (17), we have

$$\sum_{1}^{2} [\ (\ \mathbf{u}_k^{\,t}\ \mathbf{q}_k\ )\ \mathbf{u}_k + (\ \mathbf{v}_k^{\,t}\ \mathbf{q}_k\ )\ \mathbf{v}_k\ ] = \mathbf{0} \tag{46}$$

or

$$\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{0} \tag{47}$$

where

$$(\ \mathbf{u}_k^{\,t}\ \mathbf{q}_k\ )\ \mathbf{u}_k + (\ \mathbf{v}_k^{\,t}\ \mathbf{q}_k\ )\ \mathbf{v}_k = \mathbf{q}_k,\ k=1,2 \tag{48}$$

(47) is equivalent to

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0} \tag{49}$$

using (20). Fig. 5 illustrates the isotropic configuration of a COMR, where the steering links of two caster wheels are symmetric with respect to y-axis, with the centers of two caster wheels and the platform lying on the line of $y=\frac{l}{2\sqrt{3}}$. Using (21), (49) can be written as

$$d\ \mathbf{u}_1 + d\ \mathbf{u}_2 + \mathbf{s}_3 = \mathbf{0} \tag{50}$$

which yields

$$\varphi_1 = \arcsin\left(\frac{1}{2\sqrt{3}}\frac{l}{d}\right), \quad \varphi_2 = \pi - \varphi_1 \tag{51}$$

subject to

$$d \geq \frac{l}{2\sqrt{3}} \tag{52}$$

Note that the isotropic configuration does not exist if the length of the steering link is less than $\frac{l}{2\sqrt{3}}$.

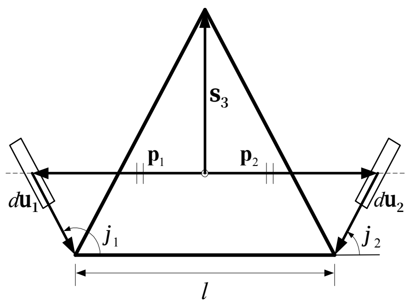

Fig. 5 Isotropic configuration for $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \varphi_2]^t$.

Finally, under the condition of (18), we have

$$\frac{1}{L^2}\sum_1^2 [\,(\mathbf{u}_k^t\ \mathbf{q}_k)^2 + (\mathbf{v}_k^t\ \mathbf{q}_k)^2\,] = 2 \tag{53}$$

or

$$\frac{1}{L^2}\sum_1^2 \|\ \mathbf{q}_k\ \|^2 = 2 \tag{54}$$

where

$$(\mathbf{u}_k^t\ \mathbf{q}_k)^2 + (\mathbf{v}_k^t\ \mathbf{q}_k)^2 = \|\ \mathbf{q}_k\ \|^2, \quad k=1,2 \tag{55}$$

With

$$\begin{aligned}\beta_1 &= \|\ \mathbf{q}_1\ \| = \|\ \mathbf{p}_1\ \| = \frac{l}{2} - d\cos(\varphi_1)\\ \beta_2 &= \|\ \mathbf{q}_2\ \| = \|\ \mathbf{p}_2\ \| = \frac{l}{2} + d\cos(\varphi_2)\end{aligned} \tag{56}$$

(54) can be written as

$$\frac{1}{L^2}(\beta_1^2 + \beta_2^2) = 2 \tag{57}$$

With (49) being held, from (57), the characteristic length of an isotropic COMR is obtained by

$$L = \beta \tag{58}$$

where

$$\beta = \beta_1 = \beta_2 \tag{59}$$

4.5 $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \theta_3]^t$

Consider the case where both rotating and steering joints of one caster wheel and two rotating joints of two caster wheels are actuated, for which $n=4$, $\sigma=2$, and $[\ \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\ ]=[\ \mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{u}_3\ ]$. First, under the condition of (16), we have

$$\mathbf{u}_1\ \mathbf{u}_1^t + \mathbf{v}_1\ \mathbf{v}_1^t + \mathbf{u}_2\ \mathbf{u}_2^t + \mathbf{u}_3\ \mathbf{u}_3^t = 2\ \mathbf{I}_2 \tag{60}$$

or

$$\mathbf{u}_2\ \mathbf{u}_2^t + \mathbf{u}_3\ \mathbf{u}_3^t = \mathbf{I}_2 \tag{61}$$

which is

$$c_2^2 + c_3^2 = 1.0, \quad c_2 s_2 + c_3 s_3 = 0.0 \tag{62}$$

The solution to (62) is obtained by

$$\varphi_3 = \varphi_2 \pm \frac{\pi}{2} \tag{63}$$

Next, under the condition of (17), we have

$$\begin{aligned}&(\mathbf{u}_1^t\ \mathbf{q}_1)\,\mathbf{u}_1 + (\mathbf{v}_1^t\ \mathbf{q}_1)\,\mathbf{v}_1\\ &\quad + (\mathbf{u}_2^t\ \mathbf{q}_2)\,\mathbf{u}_2 + (\mathbf{u}_3^t\ \mathbf{q}_3)\,\mathbf{u}_3 = \mathbf{0}\end{aligned} \tag{64}$$

or

$$\mathbf{q}_1 + (\mathbf{u}_2^t\ \mathbf{q}_2)\,\mathbf{u}_2 + (\mathbf{u}_3^t\ \mathbf{q}_3)\,\mathbf{u}_3 = \mathbf{0} \tag{65}$$

which is equivalent to

$$\mathbf{p}_1 + (\mathbf{v}_2^t\ \mathbf{s}_2)\,\mathbf{v}_2 + (\mathbf{v}_3^t\ \mathbf{s}_3)\,\mathbf{v}_3 = \mathbf{0} \tag{66}$$

using (19) and (22). Fig. 6 illustrates the isotropic configuration of a COMR, where the steering links of the second and the third caster wheels are perpendicular to each other and the center of the first caster wheel is located in such a way as to satisfy (66). Using (21), from (66), we have

$$\mathbf{u}_1 = \frac{1}{d}\,[\ \mathbf{s}_1 + (\mathbf{v}_2^t\ \mathbf{s}_2)\,\mathbf{v}_2 + (\mathbf{v}_3^t\ \mathbf{s}_3)\,\mathbf{v}_3\ ] \tag{67}$$

which restricts φ_1. Note that the isotropic configuration does not exist, unless (67) is satisfied.

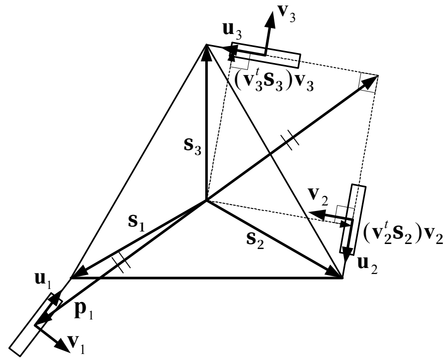

Fig. 6 Isotropic configuration for $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \theta_3]^t$.

Finally, under the condition of (18), we have

$$\begin{aligned}&\frac{1}{L^2}\,[\,(\mathbf{u}_1^t\ \mathbf{q}_1)^2 + (\mathbf{v}_1^t\ \mathbf{q}_1)^2\\ &\quad + (\mathbf{u}_2^t\ \mathbf{q}_2)^2 + (\mathbf{u}_3^t\ \mathbf{q}_3)^2\,] = 2\end{aligned} \tag{68}$$

or

$$\frac{1}{L^2}\,[\ \|\ \mathbf{q}_1\ \|^2 + (\mathbf{v}_2^t\ \mathbf{s}_2)^2 + (\mathbf{v}_3^t\ \mathbf{s}_3)^2\ ] = 2 \tag{69}$$

using (22). With (66) being held, (69) becomes

$$\frac{2\ \|\ \mathbf{p}_1\ \|^2}{L^2} = 2 \tag{70}$$

where

$$\|\ \mathbf{q}_1\ \|^2 = \|\ \mathbf{p}_1\ \|^2 = (\mathbf{v}_2^t\ \mathbf{s}_2)^2 + (\mathbf{v}_3^t\ \mathbf{s}_3)^2 \tag{71}$$

From (71), the characteristic length of an isotropic COMR is obtained by

$$L = \|\ \mathbf{p}_1\ \| \tag{72}$$

Similar analysis to the above can be made for $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \varphi_2\ \varphi_3]^t$ where both rotating and steering joints of one caster wheel and two steering joints of two caster wheels are actuated, and also for $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \varphi_3]^t$ where both rotating and steering joints of one caster wheel and one rotating and one steering joints of two caster wheels are actuated.

4.6 $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \varphi_2\ \theta_3]^t$

Consider the case where both rotating and steering joints of two caster wheels and the rotating joint of one caster wheel are actuated, for which $n=5$, $\sigma=2.5$ and $[\ \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\ \mathbf{g}_5\ ]=[\ \mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{v}_2\ \mathbf{u}_3\ ]$. First, under the condition of (16), we have

$$\sum_{k=1}^{2} (\ \mathbf{u}_k\ \ \mathbf{u}_k^{\ t} + \mathbf{v}_k\ \ \mathbf{v}_k^{\ t}\ ) + \mathbf{u}_3\ \ \mathbf{u}_3^{\ t} = 2.5\ \mathbf{I}_2 \quad (73)$$

or

$$\mathbf{u}_3\ \ \mathbf{u}_3^{\ t} = 0.5\ \mathbf{I}_2 \quad (74)$$

There does not exist φ_3 satisfying (74), and the isotropy of $\mathbf{A}$ cannot be achieved at all.

Similar analysis to the above can be made for the case of $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \varphi_2\ \varphi_3]^t$, where both rotating and steering joints of two caster wheels and the steering joint of one caster wheel are actuated.

4.7 $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \varphi_2\ \theta_3\ \varphi_3]^t$

Consider the case where both rotating and steering joints of three caster wheels are actuated, for which $n=6$, $\sigma=3$, and $[\ \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\ \mathbf{g}_5\ \mathbf{g}_6\ ]=[\ \mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{v}_2\ \mathbf{u}_3\ \mathbf{v}_3\ ]$. First, under the condition of (16), we have

$$\sum_{1}^{3} (\ \mathbf{u}_k\ \ \mathbf{u}_k^{\ t} + \mathbf{v}_k\ \ \mathbf{v}_k^{\ t}\ ) = 3\ \mathbf{I}_2 \quad (75)$$

which always holds.

Next, under the condition of (17), we have

$$\sum_{1}^{3} [\ (\ \mathbf{u}_k^{\ t}\ \mathbf{q}_k\ )\ \mathbf{u}_k + (\ \mathbf{v}_k^{\ t}\ \mathbf{q}_k\ )\ \mathbf{v}_k\ ] = \mathbf{0} \quad (76)$$

which is equivalent to

$$\sum_{1}^{3} \mathbf{p}_k = \mathbf{0} \quad (77)$$

using (20). Fig. 7 illustrates the isotropic configuration of a COMR. where the centers of three caster wheels are symmetric with respect to the center of the platform. Using (21), (77) can be written as

$$\sum_{1}^{3} \mathbf{u}_k = \mathbf{0} \quad (78)$$

where $\sum_{1}^{3} \mathbf{s}_k = \mathbf{0}$ is used. The solution to (78) is obtained by

$$\varphi_2 = \varphi_1 \pm \frac{2}{3}\pi, \quad \varphi_3 = \varphi_1 \mp \frac{2}{3}\pi \quad (79)$$

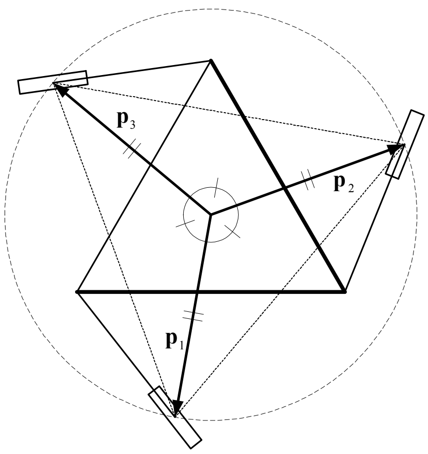

Fig. 7 Isotropic configuration for $\boldsymbol{\Theta}=[\theta_1\ \varphi_1\ \theta_2\ \varphi_2\ \theta_3\ \varphi_3]^t$.

Finally, under the condition of (18), we have

$$\frac{1}{L^2}\sum_{1}^{3} [\ (\ \mathbf{u}_k^{\ t}\ \mathbf{q}_k\ )^2 + (\ \mathbf{v}_k^{\ t}\ \mathbf{q}_k\ )^2\ ] = 3 \quad (80)$$

or

$$\frac{1}{L^2}\sum_{1}^{3} \|\ \mathbf{p}_k\ \|^2 = 3 \quad (81)$$

using (20). With

$$\begin{aligned} \beta_1^{\ 2} &= \|\ \mathbf{p}_1\ \|^2 = d^2+(\frac{l}{\sqrt{3}})^2-\frac{2dl}{\sqrt{3}}\cos(\varphi_1-\frac{\pi}{6}) \\ \beta_2^{\ 2} &= \|\ \mathbf{p}_2\ \|^2 = d^2+(\frac{l}{\sqrt{3}})^2-\frac{2dl}{\sqrt{3}}\cos(\varphi_2-\frac{5}{6}\pi) \\ \beta_2^{\ 2} &= \|\ \mathbf{p}_3\ \|^2 = d^2+(\frac{l}{\sqrt{3}})^2-\frac{2dl}{\sqrt{3}}\cos(\varphi_3+\frac{\pi}{2}) \end{aligned} \quad (82)$$

(81) can be written as

$$\frac{1}{L^2}(\ \beta_1^{\ 2}+\ \beta_2^{\ 2}+\ \beta_3^{\ 2}) = 3 \quad (83)$$

With (79) being held, from (83), the characteristic length of an isotropic COMR is obtained by

$$L = \beta \quad (84)$$

where

$$\beta = |\beta_1| = |\beta_2| = |\beta_3| \quad (85)$$

5. CONCLUSION

This paper identified the isotropic configurations of an omnidirectional mobile robot with three caster wheels. First, the kinematic model of a caster wheeled omnidirectional mobile robot(COMR) was obtained without matrix inversion. Second, with the characteristic length introduced, the isotropy conditions on two Jacobian matrices were derived. For the isotropy of one Jacobian matrix, three caster wheels should be identical with the length of the steering link equal to the radius of the wheel. Third, depending on the selection of actuated joints, the isotropy conditions on the other Jacobian matrix were derived and the isotropic configurations of a COMR were identified. The global isotropy of a COMR over entire configurations are under study to determine the optimal characteristic length as well as design parameters.

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