Feasible and Invariant Sets for Input Constrained Linear Parameter Varying Systems

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Abstract: Parameter set of an LPV system is divided into a number of subsets so that robust feedback gains may be designed for each subset of parameters. A concept of quasi-invariant set is introduced, which allows finite steps of delay in reentrance to the set. A feasible and positively invariant set with respect to a gain-scheduled state feedback control can be easily obtained from the quasi-invariant set. A receding horizon control strategy can be derived based on this feasible and invariant set.

Keywords: LPV, Input Constraints, Invariant Sets, Receding Horizon Control

1. INTRODUCTION

Receding horizon dual-mode paradigm provides an effective way to handle control problems in the presence of physical constraints on actuation[1][2][3] [5][6]. The basic idea here is to use a finite number, N, of feasible free control moves to steer a state into a target set that is feasible and positively invariant with respect to a state feedback gain K. The feasible and positively invariant set is defined as a set of states for which a state feedback control $\mathbf{u} = K\mathbf{x}$ satisfies physical constraints for any state of the set and makes it remain within the set. In the presence of model uncertainties, [5] and [8] derived polyhedral and ellipsoidal invariant sets, respectively, with respect to a feedback gain K. Existence of such a polyhedral/ellipsoidal invariant set requires that the feedback gain K robustly stabilize the uncertain system. If some information on the current system parameters is available (although their future variation is uncertain), however, we could assume the use of different gains depending on current parameters to yield better performances. In this gain-scheduled control, we do not need to find a single feedback gain which can stabilize the whole uncertainty class.

In recent years there has been several works on the gain-scheduled control of linear parameter-varying(LPV) systems utilizing invariant sets[9][10][11]. Free control moves are computed concerning parameter dependent gains but invariant sets are defined with respect to a single feedback gain in [9] and [10]. In [11], invariant sets are derived with respect to a gain-scheduled control. These works are considering ellipsoidal invariant sets such as $\{\mathbf{x}|\mathbf{x}'P\mathbf{x} \le 1\}$, where P is not parameter dependent. As it is known that parameter-dependent Lyapunov functions yield better stability results than fixed quadratic Lyapunov functions in robust control problems[12], use of parameter dependent paper, however, we seek improvement in a different way considering gain-scheduled control. Concept of quasi-invariance will be introduced, which postpone the invariance requirement until some finite future time steps *i.e.* states may leave the set but return into the set in finite time steps, and this will relieve the conservativeness of existing invariance conditions. Unlike the earlier works[9][10][11], polyhedral type invariant sets will be used in this paper. Once a quasi-invariant set is obtained, it is straight forward to extend it to yield a positively invariant polyhedral set of higher complexity.

This work was supported by Korea Research Foundation Grant (KRF-2002-041-D00206).

2. Introduction of Quasi-Invariance

Consider the following Linear Parameter Varying(LPV) system:

$$\mathbf{x}(k+1) = \tilde{A}(\theta)\mathbf{x}(k) + \tilde{B}(\theta)\mathbf{u}(k),\ |\mathbf{u}(k)| \le \bar{\mathbf{u}}, \quad (1)$$

where θ is a vector of time-varying parameter and the matrix functions $\tilde{A}(\cdot)$ and $\tilde{B}(\cdot)$ belong to one of the uncertainty class depending on the value of θ:

$$\Omega_m = \left\{ (\tilde{A}_m, \tilde{B}_m) | (\tilde{A}_m, \tilde{B}_m) = \sum_{l=1}^{n_p} \eta_l (A_{m,l}, B_{m,l}), \right.$$
$$\left. \eta_l \ge 0, \sum_{l=1}^{n_p} \eta_l = 1 \right\}, m = 1, 2, \cdots, Q. \quad (2)$$

Remark 1 :*A polyhedral set* $\Omega = \{(\tilde{A},\tilde{B})|(\tilde{A},\tilde{B}) = \sum_{l=1}^{n_s} \eta_l(A_l, B_l),\ \eta_l \ge 0, \sum_{l=1}^{n_s}\eta_l = 1$ *can be divided into subsets* $\Omega_m(m = 1,2,\cdots,Q)$, *where* $(A_{m,l}, B_{m,l})s(m = 1,2,\cdots,Q,\ l = 1,2,\cdots,n_p)$ *are composed of* $(A_l,B_l)(l = 1,2,\cdots,n_s)$ *and other elements of* Ω. *There may be infinite number of choices in dividing* Ω. *As the number of subset increases, it is likely to yield better performance while the complexity of control law increases.* ■

In the sequel, modulus of a vector/matrix and inequalities between vectors are defined as:

$$|M| := \begin{bmatrix} |m_{1,1}| & |m_{1,2}| & \cdots & |m_{1,p}| \\ |m_{2,1}| & |m_{2,2}| & \cdots & |m_{2,p}| \\ \vdots & & \cdots & \vdots \\ |m_{q,1}| & |m_{q,2}| & \cdots & |m_{q,p}| \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} \le \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} \ implies \ \begin{matrix} \alpha_1 \le \beta_1, \\ \alpha_2 \le \beta_2, \\ \vdots \\ \alpha_p \le \beta_p \end{matrix}.$$

The parameter vector θ is uncertain but its current value can be measured or estimated on-line. Thus at each time instant, it is possible to identify the uncertainty class to which $(\tilde{A}(\theta), \tilde{B}(\theta))$ belongs. A reasonable control strategy for this system is to design state feedback gains K_m

$(m = 1, 2, \cdots, Q)$ which robustly stabilize uncertainty classes Ω_m $(m = 1, 2, \cdots, Q)$, respectively and switch feedback gains according to the parameter variation. We will denote this control law as:

$$\mathbf{u}(k) = K(\theta)\mathbf{x}(k), \tag{3}$$

where $K(\theta)$ can be one of $K_m,\ m = 1, 2, \cdots, Q$ depending on the value of θ. For a given measured state $\mathbf{x}(k)$, based on the measurement of θ, one of the control law $\mathbf{u}(k) = K_m\mathbf{x}(k), m = 1, 2, \cdots, Q$ shall be used, which requires

$$|\mathbf{u}(k)| = |K_m\mathbf{x}(k)| \le \overline{\mathbf{u}},\ m = 1, 2, \cdots, Q. \tag{4}$$

Provided that (4) is satisfied, use of $\mathbf{u}(k) = K(\theta)\mathbf{x}(k)$ would yield

$$\mathbf{x}(k+1) = \tilde{\Phi}(\theta)\mathbf{x}(k),\ \ \tilde{\Phi}(\theta) := \tilde{A}(\theta) + \tilde{B}(\theta)K(\theta). \tag{5}$$

As a first step to tackle the problem of obtaining positively invariant set for the LPV system (1-2) with respect to the switching control (3), we make the following definition.

Definition : *Consider the LPV system described by (1) and (2). A set $\mathcal{F}$ is feasible and* **quasi-invariant** *with respect to the switching feedback control $\mathbf{u}(k) = K(\theta)\mathbf{x}(k)$ of (3) if there exist a finite positive number n_{inv} such that for any initial state $\mathbf{x}(k) \in \mathcal{F}$, the future states $\mathbf{x}(k+i)$ $(i = 1, \cdots, n_{inv})$ of the system (5) satisfy the input constraint (4)(feasible) and $\mathbf{x}(k+n_{inv})$ belongs to $\mathcal{F}$(quasi-invariant) at any possible variation of θ.* ■

Note that any set of states containing the origin as its interior point would be quasi-invariant provided that the system under consideration is asymptotically stable, since states of stable systems converge to the origin from any initial value. This fact shows that the quasi-invariance of a set does not much depend on the shape or complexity of the set. Thus, we will consider a simple polyhedral set and derive conditions under which the set is feasible and quasi-invariant with respect to (3).

Consider a simple polyhedral set of states which contains the origin:

$$\mathcal{F} := \left\{ \mathbf{x} | \mathbf{x} = \sum_{i=1}^{n} \{\eta_i \mathbf{x}_i + \mu_i \mathbf{x}_{i-}\},\ \eta_i, \mu_i \ge 0, \eta_i \cdot \mu_i = 0, \right.$$
$$\left. \sum_{i=1}^{n} \{\eta_i + \mu_i\} = 1 \right\}, \tag{6}$$

where

$$\mathbf{x}_i := \frac{1}{\lambda_i}\tilde{\mathbf{x}}_i \tag{7}$$

$$\mathbf{x}_{i-} := -\frac{1}{\lambda_i}\tilde{\mathbf{x}}_i, \tag{8}$$

$\lambda_i > 0$ and $\tilde{\mathbf{x}}_i$ represents the i^{th} canonical unit vector of R^n space. Note that $\mathbf{x}_i$ and $\mathbf{x}_{i-}$ are vectors of opposite directions and we assume that only one of these two vectors has nonzero coefficient. The scaling factors $\lambda_i,\ i = 1, 2, \cdots, n$ can be used to adjust the size of $\mathcal{F}$ so that the control law (3) remain feasible for any initial state in the set.

We assume that the feedback gains $K_i,\ i = 1, 2, \cdots, Q$ are pre-determined. Our aim here is to derive sufficient conditions in terms of λ_i, under which $\mathcal{F}$ is feasible and quasi-invariant with respect to (3). It is required to check whether the predicted future states $\mathbf{x}(k+i|k)$ $(i = 1, \cdots, n_{inv})$ satisfy the input constraint (4) and $\mathbf{x}(k+n_{inv}|k)$ belongs to $\mathcal{F}$ for any initial state $\mathbf{x}(k) \in \mathcal{F}$. Because of the uncertainties resides in $\tilde{\Phi}_m$, however, it is not possible to compute the exact propagation of states. Instead, upper and lower bounds on $\mathbf{x}(k+j|k)$ can be obtained using the recursive state bounding technique developed in [5]. It is based on the following obvious lemma.

Lemma : *Given a matrix M, let $M^+ := max(M, 0)$ and $M^- := max(-M, 0)$, then*

$$\underline{\mathbf{x}} \le \mathbf{x} \le \overline{\mathbf{x}} \Longrightarrow M^+\underline{\mathbf{x}} - M^-\overline{\mathbf{x}} \le M\mathbf{x} \le M^+\overline{\mathbf{x}} - M^-\underline{\mathbf{x}}, \tag{9}$$

where the maximization is applied elementwise. ■

Assume that the current state is given as

$$\mathbf{x}(k) = \tilde{\mathbf{x}}_i, \tag{10}$$

then the upper and lower bounds on $\mathbf{x}(k)$ can be determined as:

$$\overline{\tilde{\mathbf{x}}}_i(k) = \tilde{\mathbf{x}}_i = \underline{\tilde{\mathbf{x}}}_i(k). \tag{11}$$

From relations (5) and $\tilde{\Phi}_m = \sum_{l=1}^{n_p} \eta_j \Phi_{m,l}$, bounds on $\mathbf{x}(k+1|k)$ can be obtained as:

$$\overline{\tilde{\mathbf{x}}}_i(k+1|k) = \max_{\substack{m = 1, 2, \cdots, Q \\ l = 1, 2, \cdots, n_p}} \Phi_{m,l}\tilde{\mathbf{x}}_i, \tag{12}$$

$$\underline{\tilde{\mathbf{x}}}_i(k+1|k) = \min_{\substack{m = 1, 2, \cdots, Q \\ l = 1, 2, \cdots, n_p}} \Phi_{m,l}\tilde{\mathbf{x}}_i, \tag{13}$$

where $\Phi_{m,l} := A_{m,l} + B_{m,l}K_m$ and the maximization/minimization is applied elementwise. Starting from the bounds such that $\overline{\tilde{\mathbf{x}}}_i(k+1|k) \le \mathbf{x}(k+1|k) \le \underline{\tilde{\mathbf{x}}}_i(k+1|k)$, the bounds $\overline{\tilde{\mathbf{x}}}_i(k+j|k)$ and $\underline{\tilde{\mathbf{x}}}_i(k+j|k)$ for $j > 1$ can be obtained recursively as:

$$\overline{\tilde{\mathbf{x}}}_i(k+j+1|k) = \max_{\substack{m = 1, 2, \cdots, Q \\ l = 1, 2, \cdots, n_p}} \Phi^+_{m,l}\overline{\tilde{\mathbf{x}}}_i(k+j|k) - \Phi^-_{m,l}\underline{\tilde{\mathbf{x}}}_i(k+j|k) \tag{14}$$

$$\underline{\tilde{\mathbf{x}}}_i(k+j+1|k) = \min_{\substack{m = 1, 2, \cdots, Q \\ l = 1, 2, \cdots, n_p}} \Phi^+_{m,l}\underline{\tilde{\mathbf{x}}}_i(k+j|k) - \Phi^-_{m,l}\overline{\tilde{\mathbf{x}}}_i(k+j|k). \tag{15}$$

Denote the set of bounds generated by relations (10-15) with initial states $\mathbf{x}(k) = \tilde{\mathbf{x}}_i$ as:

$$\mathcal{B}(\tilde{\mathbf{x}}_i) := \{(\overline{\tilde{\mathbf{x}}}_i(k+j|k), \underline{\tilde{\mathbf{x}}}_i(k+j|k)),\ j = 0, 1, \cdots, n_{inv}\}, \tag{16}$$

and consider hypercubes defined by these bounds as

$$\mathcal{H}_j(\tilde{\mathbf{x}}_i) := \left\{ \mathbf{x} \in R^n \;\middle|\; \underline{\tilde{\mathbf{x}}}_i(k+j|k) \le \mathbf{x} \le \overline{\tilde{\mathbf{x}}}_i(k+j|k) \right\}. \tag{17}$$

Since $\mathcal{F}$ is convex and the closed-loop dynamics (5) is linear in terms of $\mathbf{x}(k)$, the feasibility and quasi-invariance of $\mathcal{F}$ can be checked by considering propagations of the vertices of $\mathcal{F}$, *i.e.* $\mathcal{B}(\mathbf{x}_i)$ and $\mathcal{H}_{n_{inv}}(\mathbf{x}_i)$ for $i = 1, 2, \cdots, n$. The quasi-invariance of $\mathcal{F}$ would be guaranteed if $\mathcal{H}_{n_{inv}}(\mathbf{x}_i) \subset \mathcal{F}$. *i.e.* every vertices of $\mathcal{H}_{n_{inv}}(\mathbf{x}_i)$ belong to $\mathcal{F}$ for $i = 1, 2, \cdots, n$.

We will denote the p^{th} vertex of $\mathcal{H}_{n_{inv}}(\tilde{\mathbf{x}}_i)$ as $\mathbf{v}^p_{n_{inv}}(\tilde{\mathbf{x}}_i)$. Each vertex of $\mathcal{H}_{n_{inv}}(\tilde{\mathbf{x}}_i)$ could be represented as a linear combination of canonical unit vectors $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{x}}_{i-}(i = 1, 2, \cdots, n)$ as:

$$\mathbf{v}^p_{n_{inv}}(\tilde{\mathbf{x}}_i) = \sum_{q=1}^{n} \{\tilde{\eta}^{p,i}_q \cdot \tilde{\mathbf{x}}_q + \tilde{\mu}^{p,i}_q \cdot \tilde{\mathbf{x}}_{q-}\}, \tag{18}$$

where $\tilde{\eta}^{p,i}_q, \tilde{\mu}^{p,i}_q \geq 0$ and $\tilde{\eta}^{p,i}_q \cdot \tilde{\mu}^{p,i}_q = 0$. Note that the coefficients $\tilde{\eta}^{p,i}_q$ and $\tilde{\mu}^{p,i}_q$ can be computed off-line for $p = 1, 2, \cdots, 2^n$ and $i = 1, 2, \cdots, n$, since the bounds $\underline{\tilde{\mathbf{x}}}_i(k + n_{inv}|k)$ and $\overline{\tilde{\mathbf{x}}}_i(k + n_{inv}|k)$ are obtained with a known initial state $\tilde{\mathbf{x}}_i$ as (10).

From the linearity of the dynamics (5) and the definitions of (7-8), we have $\mathbf{v}^p_{n_{inv}}(\mathbf{x}_i) = \frac{1}{\lambda_i}\mathbf{v}^p_{n_{inv}}(\tilde{\mathbf{x}}_i)$. Multiplying $\frac{1}{\lambda_i}$ on both sides of (18), we have:

$$\mathbf{v}^p_{n_{inv}}(\mathbf{x}_i) = \sum_{q=1}^{n} \{\tilde{\eta}^{p,i}_q \cdot \frac{\lambda_q}{\lambda_i} \cdot \mathbf{x}_q + \tilde{\mu}^{p,i}_q \cdot \frac{\lambda_q}{\lambda_i} \cdot \mathbf{x}_{q-}\}. \tag{19}$$

It is easy to see that $\mathbf{v}^p_{n_{inv}}(\mathbf{x}_i)$ would belong to $\mathcal{F}$ provided that:

$$\sum_{q=1}^{n} \lambda_q \cdot \tilde{\eta}^{p,i}_q + \lambda_q \cdot \tilde{\mu}^{p,i}_q = \lambda_i. \tag{20}$$

It is clear that every element of the hypercube $\mathcal{H}_{n_{inv}}(\mathbf{x}_i)$ would be included in the set $\mathcal{F}$ if and only if all the vertices of $\mathcal{H}_{n_{inv}}(\mathbf{x}_i)$ belong to $\mathcal{F}$. Thus, the quasi-invariance of the set $\mathcal{F}$ would be established if there exist positive $\lambda_i s$ satisfying the relation (20) for $p = 1, 2, \cdots, 2^n$ and $i = 1, 2, \cdots, n$.

On the other hand, the bounds on $K_m\mathbf{x}(k + j|k)(m = 1, 2, \cdots, Q)$, where $\mathbf{x}(k + j|k)$ is obtained with initial state $\mathbf{x}(k) = \mathbf{x}_i(k)$, also can be obtained using the bounds $(\overline{\tilde{\mathbf{x}}}_i(k + j|k),\ \underline{\tilde{\mathbf{x}}}_i(k + j|k)) \in \mathcal{B}(\tilde{\mathbf{x}}_i)$ and their feasibilities are guaranteed if conditions:

$$\begin{aligned} K^+_m\underline{\tilde{\mathbf{x}}}_i(k + j|k) - K^-_m\overline{\tilde{\mathbf{x}}}_i(k + j|k) &\geq -\lambda_i \cdot \overline{\mathbf{u}}, & (21)\\ K^+_m\overline{\tilde{\mathbf{x}}}_i(k + j|k) - K^-_m\underline{\tilde{\mathbf{x}}}_i(k + j|k) &\leq \lambda_i \cdot \overline{\mathbf{u}}, & (22) \end{aligned}$$

are satisfied for $m = 1, 2, \cdots, Q$.

Based on the above arguments, conditions for the feasibility and quasi-invariance of the set $\mathcal{F}$ can be summarized as per the following theorem.

Theorem 1: Consider the LPV system (1-2) with the switching control (3). Bounds $(\underline{\tilde{\mathbf{x}}}_i(k + j|k), \overline{\tilde{\mathbf{x}}}_i(k + j|k) \in \mathcal{B}(\tilde{\mathbf{x}}_i)(j = 1, 2, \cdots, n_{inv})$ are generated according to the recursive relations (10-15) and coefficients $\tilde{\eta}^{p,i}_q$, $\tilde{\mu}^{p,i}_q$ of (18) are obtained for vertices of the hypercube $\mathcal{H}_{n_{inv}}(\tilde{\mathbf{x}}_i)$ of (17). If the positive coefficients $\lambda_i(i = 1, 2, \cdots, n)$ satisfy (20) and (21-22) for $p = 1, 2, \cdots, 2^n$ and $i = 1, 2, \cdots, n$, then the set $\mathcal{F}$ of (6) is feasible and quasi-invariant with respect to the switching control (3). ■

Note that the relations (20) and (21-22) are linear in terms of $\lambda_i(i = 1, 2, \cdots, n)$. We can formulate an LP problem to obtain a set of coefficients satisfying these relations. One possible choice of a cost index for the LP problem would be sum of λ_is, minimizing this sum will increase the size of $\mathcal{F}$.

3. Use of Transformed States

The bounds developed according to relations (10-15), however, are likely to be overly conservative because of two reasons. The first source of conservativeness is the fact that a single set of bounds is obtained considering all the possible variation of dynamics at each future time steps. A remedy for this would be the use of different sets of bounds for different possible trajectories of future system changes. For a given current state $\mathbf{x}(k)$, we can obtain Q different sets of bounds on $\mathbf{x}(k + 1|k)$ considering the possible uncertainty sets $\Omega_m,\ m = 1, 2, \cdots, Q$. Based on the Q different sets of bounds on $\mathbf{x}(k+1|k)$, one possible way to proceed to the next time step $k+2$ would be generating Q different sets of bounds for each set of previous bounds on $\mathbf{x}(k + 1|k)$ *i.e.* generate Q^2 sets of bounds on $\mathbf{x}(k + 2|k)$. Following this strategy, we would have Q^j sets of bounds on the state $\mathbf{x}(k + j|k)$.

The second reason of conservativeness is that the bounds generated by the relations (12-15) might not be shrinking even if all the dynamics $\Phi_{m,l}(m = 1, 2, \cdots, Q,\ l = 1, 2, \cdots, n_p)$ are stable. This conservativeness can be relieved by introducing state transformations $\mathbf{z}^i := W_i\mathbf{x}$ so that the bounds on the transformed states can be made shrinking [5] *i.e.* $\underline{\mathbf{z}}^i(k + j|k) \leq \underline{\mathbf{z}}^i(k + j + 1|k)$ and $\overline{\mathbf{z}}^i(k + j + 1|k) \leq \overline{\mathbf{z}}^i(k + j|k)$.

Consider Q different state transformation $W_m\mathbf{x}, (m = 1, 2, \cdots, Q)$, where W_m represents the transformation matrix corresponding to a state feedback gain K_m. Using these transformed states, relations (10) and (12-13) can be rewritten as:

$$\mathbf{z}^m(k) = W_m\tilde{\mathbf{x}}_i \tag{23}$$

and

$$\overline{\tilde{\mathbf{z}}}^m_i(k + 1|k) = \max_{l = 1, 2, \cdots, n_p} \Phi^{W_m}_{m,l} W_m\tilde{\mathbf{x}}_i \tag{24}$$

$$\underline{\tilde{\mathbf{z}}}^m_i(k + 1|k) = \min_{l = 1, 2, \cdots, n_p} \Phi^{W_m}_{m,l} W_m\tilde{\mathbf{x}}_i, \tag{25}$$

where and $\Phi^{W_m}_{m,l} := W_m(A_{m,l} + B_{m,l}K_m)W^{-1}_m$. Based on the Q different sets of upper and lower bounds $\overline{\tilde{\mathbf{z}}}^m_i(k + 1|k)$, $\underline{\tilde{\mathbf{z}}}^m_i(k + 1|k)$ $(m = 1, 2, \cdots, Q)$, we will generate Q^2 different sets of bounds $\overline{\tilde{\mathbf{z}}}^{\hat{m}}_i(k + 2|k)$, $\underline{\tilde{\mathbf{z}}}^{\hat{m}}_i(k + 2|k)(\hat{m} = 1, 2, \cdots, Q^2)$, .

A set of states between the bounds $\underline{\tilde{\mathbf{z}}}^{\hat{m}}(k + j|k)$ and $\overline{\tilde{\mathbf{z}}}^{\hat{m}}(k + j|k)$ can be defined as:

$$\mathcal{H}_j(\hat{m}, \tilde{\mathbf{x}}_i) := \left\{\mathbf{z} \in R^n | \underline{\tilde{\mathbf{z}}}^{\hat{m}}_i(k + j|k) \leq \mathbf{z} \leq \overline{\tilde{\mathbf{z}}}^{\hat{m}}_i(k + j|k)\right\} \tag{26}$$

then Q^{j+1} different set of bounds based on Q^j different set of bounds can be obtained as:

$$\bar{\tilde{\mathbf{z}}}_i^{\check{m}}(k+j+1|k) = \max_{\substack{l=1,2,\cdots,n_p \\ p=1,2,\cdots,2^n}} \Phi_{m.l}^{W_m} W_m W_{r(\frac{\hat{m}}{Q})}^{-1} \mathbf{v}_j^p(\check{m},\tilde{\mathbf{x}}_i) \quad (27)$$

$$\underline{\tilde{\mathbf{z}}}_i^{\check{m}}(k+j+1|k) = \min_{\substack{l=1,2,\cdots,n_p \\ p=1,2,\cdots,2^n}} \Phi_{m.l}^{W_m} W_m W_{r(\frac{\hat{m}}{Q})}^{-1} \mathbf{v}_j^p(\check{m},\tilde{\mathbf{x}}_i) \quad (28)$$

for $m=1,2,\cdots,Q$, $\check{m}=1,2,\cdots,Q^j$ and $\hat{m}:=(\check{m}-1)Q+m$. As it was done for non-transformed case, each vertex of $\mathcal{H}_{n_{inv}}(\hat{m},\tilde{\mathbf{x}}_i)$ can be represented as a linear combination of transformed canonical unit vectors:

$$\mathbf{v}_{n_{inv}}^p(\hat{m},\mathbf{x}_i) = \quad (29)$$
$$\sum_{q=1}^{n}\{\tilde{\eta}_q^{p,i,\hat{m}} \cdot \frac{\lambda_q}{\lambda_i} W_{r(\frac{\hat{m}}{Q})}\mathbf{x}_q + \tilde{\mu}_q^{p,i,\hat{m}} \cdot \frac{\lambda_q}{\lambda_i} W_{r(\frac{\hat{m}}{Q})}\mathbf{x}_{p-}\},$$

where $\tilde{\eta}_q^{p,i,\hat{m}}$, $\tilde{\mu}_q^{p,i,\hat{m}} \geq 0$ and $\tilde{\eta}_q \cdot \tilde{\mu}_q = 0$.
It is easy to see that $\mathcal{F}$ of (6) can be rewritten in terms of transformed states as:

$$\mathcal{F} = \left\{\mathbf{x}|W_m\mathbf{x} = \sum_{q=1}^{n}\{\eta_i W_m \mathbf{x}_i + \mu_i W_m \mathbf{x}_{i-}\},\ \eta_i,\mu_i \geq 0,\right.$$
$$\left.\eta_i \cdot \mu_i = 0, \sum_{i=1}^{n}\{\eta_i+\mu_i\} \leq 1\right\}. \quad (30)$$

From (29) and (30), the scaled vertex $\mathbf{v}_{n_{inv}}^p(\hat{m},\mathbf{x}_i)$ belongs to $\mathcal{F}$ if

$$\sum_{q=1}^{n}\lambda_q\tilde{\eta}_q^{p,i,\hat{m}} + \lambda_q\tilde{\mu}_q^{p,i,\hat{m}} \leq \lambda_i \qquad \begin{array}{c} p=1,2,\cdots,2^n \\ i=1,2,\cdots,n \\ \hat{m}=1,2,\cdots,Q^{n_{inv}} \end{array} \quad (31)$$

is satisfied.
Additional conditions which guarantee the feasibility of state feedback controls during time steps k to $k+n_{inv}-1$ can be obtained in terms of the transformed states as:

$$\mathcal{K}_{m,r(\frac{\hat{m}}{Q})}^{+}\underline{\tilde{\mathbf{z}}}_i^{\check{m}}(k+j|k) - \mathcal{K}_{m,r(\frac{\hat{m}}{Q})}^{-}\bar{\tilde{\mathbf{z}}}_i^{\check{m}}(k+j|k) \geq -\lambda_i\cdot\bar{\mathbf{u}}, \quad (32)$$
$$\mathcal{K}_{m,r(\frac{\hat{m}}{Q})}^{+}\bar{\tilde{\mathbf{z}}}_i^{\hat{m}}(k+j|k) - \mathcal{K}_{m,r(\frac{\hat{m}}{Q})}^{-}\underline{\tilde{\mathbf{z}}}_i^{\hat{m}}(k+j|k) \leq \lambda_i\cdot\bar{\mathbf{u}}, \quad (33)$$

for $j=0,1,\cdots,n_{inv}-1$, $\check{m}=1,2,\cdots,Q^j$,where $\mathcal{K}_{\hat{m}}^{+} := max(K_{\hat{m}}W_{\hat{m}}^{-1},0)$ and $\mathcal{K}_{\hat{m}}^{-} := max(-K_{\hat{m}}W_{\hat{m}}^{-1},0)$
From the definitions (7-8), we can see that minimizing λ_is would yield $\mathbf{x}_i$s of maximum length and in turn the size of $\mathcal{F}$ would be maximized. Based on this observation, it is possible to formulate an LP problem obtaining optimal λ_is which guarantee the quasi-invariance of $\mathcal{F}$:

$$[\lambda_1^* \ \lambda_2^* \ \cdots \ \lambda_n^*] = \arg\min_{\lambda_1,\cdots,\lambda_n} \sum_{i=1}^{n}\lambda_i \quad (34)$$

subject to (31) and (32-33).

4. Positively Invariant Sets

Assume that the set $\mathcal{F}$ of (6) is quasi-invariant and the corresponding hypercubes $\mathcal{H}_j(\hat{m},\tilde{\mathbf{x}}_i)$ ($j = 1,2,\cdots,n_{inv}$, $\hat{m}=1,2,\cdots,Q^j$, $i=1,2,\cdots,n$) of (??) and positive coefficients $\lambda_i(i=1,2,\cdots,n)$ of (34) are obtained, then we have:

$$\mathcal{H}_j(\hat{m},\mathbf{x}_i) = \frac{1}{\lambda_i}\mathcal{H}_j(\hat{m},\tilde{\mathbf{x}}_i) \quad (35)$$
$$\mathbf{v}_j^p(\hat{m},\mathbf{x}_i) = \frac{1}{\lambda_i}\mathbf{v}_j^p(\hat{m},\tilde{\mathbf{x}}_i). \quad (36)$$

Note that state feedback laws $\mathbf{u}=K_m\mathbf{x}(m=1,2,\cdots,Q)$ are feasible for any $\mathbf{x}\in\mathcal{H}_j(\hat{m},\mathbf{x}_i)$ and $\mathbf{x}\in\mathcal{F}$. From this fact and the feasibility/quasi-invariance of $\mathcal{F}$, it is possible to establish a feasible and positively invariant convex hull, composed of $\mathcal{H}_j(\hat{m},\mathbf{x}_i)$ and $\mathcal{F}$, as per the following theorem.

Theorem 2: Consider the LPV system (1-2) with the switching control (3). Bounds $(\bar{\tilde{\mathbf{z}}}_i^m(k+j|k), \underline{\tilde{\mathbf{z}}}_i^m(k+j|k))$ are generated according to the recursive relations (24-28) and the hypercubes $\mathcal{H}_j(\hat{m},\mathbf{x}_i)$ ($j=1,2,\cdots,n_{inv}$, $\hat{m}=1,2,\cdots,Q^j$, $i=1,2,\cdots,n$) and their vertices $\mathbf{v}_j^p(\hat{m},\mathbf{x}_i)(p=1,2,\cdots,2^n)$ are obtained as (35-36). If the positive coefficients $\lambda_i(i=1,2,\cdots,n)$ satisfy (31) and the bounds $(\bar{\tilde{\mathbf{z}}}_i^m(k+j|k), \underline{\tilde{\mathbf{z}}}_i^m(k+j|k))$ satisfy (32-33), then the convex hull defined as:

$$\mathcal{C} := \left\{\mathbf{x}|\mathbf{x} = \sum_{i=1}^{n}[\eta_i\mathbf{x}_i + \mu_i\mathbf{x}_{i-}]\right. \quad (37)$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n_{inv}-1}\sum_{\hat{m}=1}^{Q^j}\sum_{p=1}^{2^n}[\zeta_i^{j,\hat{m},p}\mathbf{x}_i^{j,\hat{m},p} + \xi_i^{j,\hat{m},p}\mathbf{x}_{i-}^{j,\hat{m},p}],$$
$$\sum_{i=1}^{n}[\eta_i+\mu_i] + \sum_{i=1}^{n}\sum_{j=1}^{n_{inv}-1}\sum_{\hat{m}=1}^{Q^j}\sum_{p=1}^{2^n}[\zeta_i^{j,\hat{m},p} + \xi_i^{j,\hat{m},p}] = 1,$$
$$\left.\eta_i,\mu_i,\zeta_i^{j,\hat{m},p},\xi_i^{j,\hat{m},p} \geq 0\right\}$$

is feasible and positively invariant with respect to the switching control (3), where $\mathbf{x}_i^{j,\hat{m},p} := W_{r(\frac{\hat{m}}{Q})}^{-1}\mathbf{v}_j^p(\hat{m},\mathbf{x}_i)$ and $\mathbf{x}_{i-}^{j,\hat{m},p} := W_{r(\frac{\hat{m}}{Q})}^{-1}\mathbf{v}_j^p(\hat{m},\mathbf{x}_{i-})$.

■

The maximum number of vertices of $\mathcal{C}$ is $2n+2(n\times(n_{inv}-1)\times Q^j\times 2^n)$ considering $\mathbf{x}_i, \mathbf{x}_{i-}, \mathbf{x}_i^{j,\hat{m},p}$ and $\mathbf{x}_{i-}^{j,\hat{m},p}$ for $i=1,2,\cdots,n$, $j=1,2,\cdots,n_{inv}-1$, $\hat{m}=1,2,\cdots,Q^j$, $p=1,2,\cdots,2^n$. Some of these points, however, might be located inside of $\mathcal{C}$ and the actual number of vertices of $\mathcal{C}$ would be less than the maximum number of vertices. If the number of actual vertices is n_{ver} and we denote the actual vertices of $\mathcal{C}$ as $\mathbf{x}_i^o$ ($i=1,2,\cdots,n_{ver}$), then $\mathcal{C}$ can be rewritten as:

$$\mathcal{C}^o = \{\mathbf{x}\in R^n|\mathbf{x} = \sum_{i=1}^{n_{ver}}[\eta_i^o\mathbf{x}_i^o + \mu_i^o\mathbf{x}_{i-}^o],$$
$$\sum_{i=1}^{n_{ver}}[\eta_i^o+\mu_i^o] = 1,\ \ \eta_i^o,\mu_i^o \geq 0\}. \quad (38)$$

Asymptotic stability of the switching control system can be checked by considering the vertices of $\mathcal{C}^o$ as per the following corollary.

Corollary 2.1 : Consider the convex set $\mathcal{C}^o$ of (38) and the switching control (3) for the LPV system (1-2). If there exist a positive scalar $\rho(<1)$ such that

$$\Phi_{m,l}\mathbf{x}_i^o \in \mathcal{C}^o(\rho), \quad (39)$$
$$for \quad m=1,2,\cdots,Q, \; l=1.2.\cdots,n_p \; i=1,2,\cdots,n_{ver},$$

then the resulting closed-loop system (5) is asymptotically stable, where $\mathcal{C}^o(\rho) := \{\mathbf{x} \in R^n|\mathbf{x} = \sum_{i=1}^{n_{ver}} \rho[\eta_i^o\mathbf{x}_i^o + \mu_i^o\mathbf{x}_{i-}^o], \sum_{i=1}^{n_{ver}}[\eta_i^o+\mu_i^o] = 1, \;\; \eta_i^o,\mu_i^o \geq 0\}$. ■

5. Numeric Example

Consider a LPV system (1) whose matrix functions $\tilde{A}(\theta)$ and $\tilde{B}(\theta)$ belong to one of the uncertainty classes of (2), where $Q=2, \; n_p=2, \; \overline{\mathbf{u}}=5$ and

$$A_{11} = \begin{bmatrix} 6.5622 & 7.0874 \\ 7.0874 & 7.7298 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 6.9764 & 7.5272 \\ 7.5272 & 8.1922 \end{bmatrix},$$
$$B_{11} = [1.2225 \; 1.1715]', \; B_{12} = [1.2723 \; 1.2260]'$$
$$A_{21} = \begin{bmatrix} 12.0896 & 12.9275 \\ 12.9275 & 13.8643 \end{bmatrix}, \; A_{22} = \begin{bmatrix} 12.8495 & 13.7271 \\ 13.7271 & 14.7032 \end{bmatrix},$$
$$B_{21} = [1.8498 \; 1.8485]', \; B_{22} = [1.9312 \; 1.9354]'.$$

Stabilizing feedback gains for the uncertainty sets Ω_1 and Ω_2 are obtained as $K_1 = [5.7419 \; 6.2291]$ and $K_2 = [6.8183 \; 7.2982]$, respectively, by solving the LQR problems for the center of gravities of each of the uncertainty sets. It is easy to see that use of K_1 or K_2 alone can not stabilize both of the uncertainty sets. The transformations matrices are selected as:

$$W_1 = \begin{bmatrix} -18.4767 & -18.5853 \\ -17.7541 & -19.2768 \end{bmatrix},$$
$$W_2 = \begin{bmatrix} -33.4799 & -34.3830 \\ -32.7561 & -35.0731 \end{bmatrix}.$$

The LP problem (34) searching for the weights $\lambda_i(i = 1,2,\cdots,n)$ becomes feasible with $n_{inv} = 3$ and its solution is $\lambda_1 = 1.3637, \; \lambda_2 = 1.4596$. The resulting quasi-feasible/invariant set is shown in Fig.1 along with propagation of the initial state $x_2 = [0 \;\; 1]'$. We can see that the predicted state may go out of the quasi-feasible/invariant set but come into the set in 3 time steps. A feasible and invariant set is obtained considering the vertices of upper and lower bounds for the prediction horizon $[t+1, t+2]$ as was shown in Fig.2.

6. Conclusions

The quasi-invariance requires that the closed-loop propagations of states of a set come into itself again in a finite number of steps while positive invariance requires that all the future propagations should belong to the set. In this sense, quasi-invariance is a more general concept than the positive invariance and it can be obtained through LP for uncertain switching systems. A positively invariant set can be obtained from a quasi-invariant set by considering the propagations of the vertices of the quasi-invariant set and the positive invariant set could have much more corners than the quasi-invariant set.

ACKNOWLEDGEMENTS

This work was supported by Korea Research Foundation Grant (KRF-2002-041-D00206).

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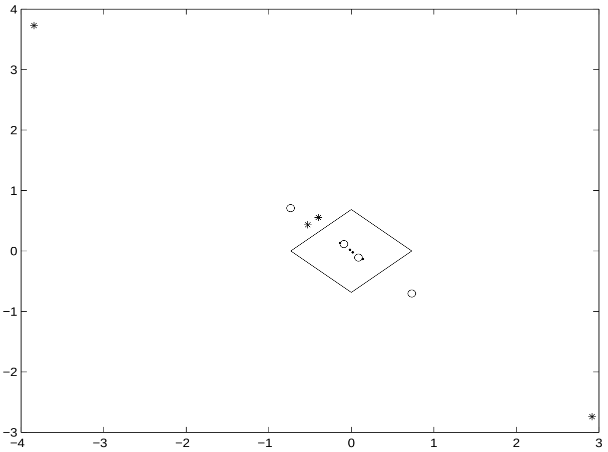

Fig. 1. Quasi-Invariant set and the vertices of possible propagation of the states with initial state $x_2 = [0\ 1]'$, '*' time step $k+1$, 'o' time step $k+2$, '.' time step $k+3$

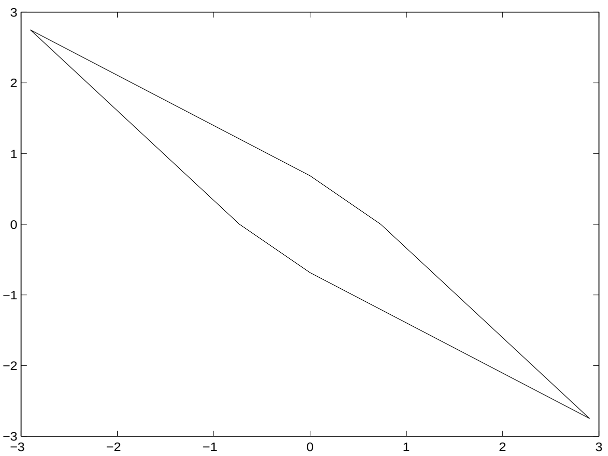

Fig. 2. Feasible and Invariant set