# Constrained MPC for uncertain time-delayed systems

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**Abstract:** It is well known that parameter uncertainties and time-delays cannot be avoided in practice and result in poor performance and even instability. Nevertheless, to the authors' best knowledge, there exist few results on model predictive control (MPC) handling explicitly uncertain time-delayed systems. In this paper, we present an MPC algorithm for uncertain time-varying systems with input constraints and state-delay. An optimization problem is suggested to find a memoryless state-feedback MPC law and the closed-loop stability is established under feasibility and certain conditions.

**Keywords:** model predictive control, uncertainty, state delay, input constraints, linear matrix inequality, closed-loop stability

## 1. Introduction

Model predictive control (MPC), also known as receding horizon control (RHC), has received much attention in control societies because of its capability of handling both constraints and time-varying behaviors and also of its good tracking performance [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. Moreover, it has been recognized as a successful control strategy in industry fields, especially in chemical process control such as petrochemical, pulp and paper control. The basic concept of MPC is to solve an optimization problem over a fixed number of future time instant at the current time and to implement the first optimal control law as the current control law. The procedure is then repeated at the next time.

It is well known that parameter uncertainties and time-delays cannot be avoided in practice, especially, in the most chemical process with slow dynamics where MPC is mainly applied. Since the parameter uncertainties and time-delays are frequently the main cause of performance degradation and instability, there has been increasing interest in the robust control of uncertain time-delay systems in other control areas, e.g. guaranteed cost control [11], [12], [13], [14], [15], [16]. However, until now, there exist few results on MPC handling explicitly uncertain time-delay the systems in the literature. Some paper considered the problem but not explicitly. [6] among them deal with the problem the most explicitly. In [6], after designing mainly the novel robust constrained MPC for uncertain systems, the authors argue that the control scheme can be extended to systems with delays in a straightforward manner. Of course, this is true when the delay indices are known. However, *when the delay indices are unknown*, it is not straightforward and not easy to show the feasibility of the on-line optimization problem and to guarantee the closed-loop stability which is the main topic of the present paper.

In this paper, we present an MPC algorithm for uncertain time-varying systems with input constraints and state-delay. The uncertainty is assumed to be the type of polytopic uncertainty, and the delay is unknown but its upper bound is assumed to be known in advance for practical reason. To find memoryless state-feedback MPC law, an optimization problem which minimizes a cost function at each sampling time is considered. After finding the upper bound of the cost function by assuming certain inequality, the original problem is relaxed to another optimization problem from which we derive the final form of optimization problem involving linear matrix inequalities (LMIs). It is shown that a feasible solution to the problem at time $k$ is also a feasible solution to the same problem after the time $k$ under certain conditions. Finally, based on the feasibility and optimality, the closed-loop stability is established.

The paper is organized as follows. Section 2. states target systems, assumptions, the associated problem. Section 3. supplies optimization problem involving LMIs and under certain conditions, the feasibility and the closed-loop stability is established. Section 4illustrates the performance of the proposed controller through an example. Finally in Section 5.we make some concluding remarks.

**Notation:** Notations in this paper are fairly standard. $\mathbb{R}^{\ltimes}$ and $\mathbb{R}^{\ltimes\times\gtrdot}$ denote the $n$-dimensional Euclidean space and the set of all $n \times m$ matrices, respectively. The notation $X \geq Y$ and $X > Y$ where $X$ and $Y$ are symmetric matrices means that $X - Y$ is positive semi-definite and positive definite, respectively. Inequalities between vectors mean component-wise inequalities. Finally $||x||_W$ denotes $x^T W x$.

## 2. Problem Statements

Consider the following discrete-time systems

$$x(k+1) = A(k)x(k) + \bar{A}(k)x(k-d) + B(k)u(k), \quad x(k) = \phi(k), \quad k \in [-d^*, 0] \tag{1}$$

subject to input constraint

$$-\bar{u} \leq u(k) \leq \bar{u}, \quad \bar{u} \geq 0, \quad \text{for all } k \in [0, \infty), \tag{2}$$

where $x(k) \in \mathbb{R}^{\ltimes}$ is the state, $u(k) \in \mathbb{R}^{\gtrdot}$ is the control, $\phi(k)$ is the initial condition, $d$ is an unknown constant integer representing the number of delay units in the state, but being assumed $0 \leq d \leq d^*$ with a known integer $d^*$. It is assumed that the system matrices $[A(k)\ \bar{A}(k)\ B(k)]$ are unknown but belongs to a polytope $\Omega$ for all times,. That is,

$$[A(k)\ \bar{A}(k)\ B(k)] \in\ \Omega \triangleq \mathrm{Co}\{[A_1\ \bar{A}_1\ B_1], ..., [A_p\ \bar{A}_p\ B_p]\}, \tag{3}$$

where Co denotes the convex hull and $[A_i\ \bar{A}_i\ B_i]$ are vertices of the convex hull. It is assumed that the system (1) is stabilizable for the existence of a stabilizing feedback control, and that the state $x(k)$ is available at each time $k$.

The goal of this paper is to find a stabilizing state-feedback control $u(k) = K(k)x(k)$ for (1) by the MPC strategy. At each time $k$, we solve the following min-max problem

$$\min_{u(\cdot|k)} \max_{[A(k)\ \bar{A}(k)\ B(k)]\in\ \Omega} J(k) \tag{4}$$

subject to

$$J(k) = \sum_{j=0}^{\infty} \{||x(k+j|k)||_Q + ||u(k+j|k)||_R\}, \tag{5}$$

$$-\bar{u} \le u(k+j|k)(= K(k)x(k+j|k)) \le \bar{u}, \quad j \in [0, \infty), \tag{6}$$

where $Q > 0$ and $R > 0$ are given symmetric matrices, and $x(k+j|k)$ and $u(k+j|k)$ denote predicted variables of the state and the input, respectively, with $x(k|k) = x(k)$.

Before ending this section, we present a fact which will be used in the next section.

Fact 21: Let us consider the following two optimization problems

$$(Q_1, X_1, Y_1) = \arg \min_{Q,X,Y} ||x||_Q + \alpha$$
$$\text{s.t. } 0 \ge F_1(Q, X, Y), ..., 0 \ge F_n(Q, X, Y),$$
$$(Q_2, X_2, Y_2) = \arg \min_{Q,X,Y} ||x||_Q + \beta$$
$$\text{s.t. } 0 \ge F_1(Q, X, Y), ..., 0 \ge F_n(Q, X, Y),$$

where $Q, X, Y$ denote optimization variables, $\alpha$ and $\beta$ denote constant terms, and $F_i(Q, X, Y)$ denotes a function of $Q, X, Y$. Suppose that the difference of the two optimization problems is only $\alpha$ and $\beta$. If one of the problems is solvable, so is the other. Moreover, the solutions of the two problems are identical, that is, $Q_1 = Q_2,\ X_1 = X_2$ and $Y_1 = Y_2$. ∎

## 3. Stabilizing MPC for delayed systems with constraints

Following an approach given in [6], it is easy to derive an upper bound on the worst value of the cost. Consider a quadratic function

$$V(k+j|k) = ||x(k+j|k)||_{P(k)} + \sum_{i=1}^{d} ||x(k+j-i|k)||_{P_d(k)}, \tag{7}$$

where $P(k) > 0,\ P_d(k) > 0$. If there exist $X(k) \triangleq \gamma(k)P^{-1}(k) > 0,\ X_d(k) \triangleq \gamma(k)P_d^{-1}(k) > 0,\ Y(k) = K(k)X(k)$ and $\gamma(k) > 0$ satisfying the following inequalities

$$\begin{bmatrix} X(k) & (*) & (*) & (*) & (*) & (*) \\ 0 & X_d(k) & (*) & (*) & (*) & (*) \\ M_{31} & \bar{A}_\sigma X_d(k) & X(k) & (*) & (*) & (*) \\ X(k) & 0 & 0 & X_d(k) & (*) & (*) \\ Q^{1/2}X(k) & 0 & 0 & 0 & \gamma(k)I & (*) \\ R^{1/2}Y(k) & 0 & 0 & 0 & 0 & \gamma(k)I \end{bmatrix} \le 0, \tag{8}$$

where $M_{31} = A_\sigma X(k) + B_\sigma Y(k)$ and $\sigma = 1, ..., p$, then the worst value of the cost $J(k)$ is bounded by

$$\max_{[A(k)\ \bar{A}(k)\ B(k)]\in\ \Omega} J(k) \le V(k|k), \tag{10}$$

where

$$V(k|k) \triangleq ||x(k|k)||_{P(k)} + \sum_{i=1}^{d} ||x(k-i|k)||_{P_d(k)}.$$

### 3.1. MPC algorithm

Based on (8) and (10), the original min-max problem (4) can be redefined to the following optimization problem that minimizes the upper bound on the worst value of the original cost function $J(k)$ :

$$\mathcal{P}_1 : \quad \min_{K(k),P(k),P_d(k)} V(k|k) \ \text{ s.t. (6), (8).} \tag{11}$$

Unfortunately, we cannot solve the problem $\mathcal{P}_1$ directly since $V(k|k)$ includes the term $\sum_{i=1}^{d} ||x(k-i|k)||_{P_d(k)}$, which depends on the unknown constant $d$. Hence, we consider the following optimization problem, which shows us an indirect way to obtain the solution (or the optimizer) of the problem $\mathcal{P}_1$.

$$\mathcal{P}_2 : \quad \min_{K(k),P(k),P_d(k)} V_u(k|k) \ \text{ s.t. (6), (8)} \tag{12}$$

where $V_u(k|k) = ||x(k|k)||_{P(k)} + \sum_{i=1}^{d^*} ||x(k-i|k)||_{P_d(k)}$. Here, we make an assumption that is effective *till the end of this subsection.*

**A1:** The $P_d(k)$ of both $\mathcal{P}_1$ and $\mathcal{P}_2$ are fixed to a constant matrix $\bar{P}_d$ for all times $k$.

Under the assumption A1, we shall present two lemmas related to the property and the feasibility of $\mathcal{P}_1$ and $\mathcal{P}_2$ and one theorem for closed-loop stability.

Lemma 31: (Equivalence of $\mathcal{P}_1$ and $\mathcal{P}_2$) If $\mathcal{P}_2$ is solvable at time $k$, then so is $\mathcal{P}_1$, and vice versa. Moreover, feasible solutions of the two problems are identical. Therefore, we can obtain the solution of $\mathcal{P}_1$ by solving $\mathcal{P}_2$.

**Proof:** Since $\bar{P}_d$ is fixed and $x(k-i|k), i = 1, ..., d^*$ are known a priori, $V(k|k)$ and $V_u(k|k)$ can be simply represented by $V(k|k) = ||x(k|k)||_{P(k)} + \alpha$ and $V(k|k) = ||x(k|k)||_{P(k)} + \beta$ respectively, where $\alpha$ and $\beta$ are known constant terms. Since the difference of the two problems is only $\alpha$ and $\beta$, by Fact 2.1, we can conclude that if one of the problems $\mathcal{P}_1$ and $\mathcal{P}_2$ is feasible at time $k$, then the other is feasible at time $k$, and that feasible solutions of the two problems are identical. ∎

Lemma 32: (Feasibility of $\mathcal{P}_1$ and $\mathcal{P}_2$) Any feasible solution of $\mathcal{P}_2$ at time $k$ is also a feasible solution of both $\mathcal{P}_1$ and $\mathcal{P}_2$ for all times greater than $k$. Thus, if the problem $\mathcal{P}_2$ is feasible at time $k$, both $\mathcal{P}_1$ and $\mathcal{P}_2$ are feasible for all times greater than $k$.

**Proof:** Suppose that a feasible control sequence $u(k+j|k), j \geq 1$ exists in $\mathcal{P}_2$ at time $k$. Then, at the next time $k+1$, choose the following control sequence

$$u(k+1+j|k+1) = u^*(k+1+j|k), \quad j \geq 0. \tag{13}$$

The control sequence satisfies the input constraint (6) at time $k+1$, which implies the problem $\mathcal{P}_2$ has a feasible solution at time $k+1$. Hence, by induction, if the problem $\mathcal{P}_2$ is feasible at time $k$, we observe that the problem is always feasible for all times greater than $k$. Moreover, by Lemma 3.1, $u(k+j|k), j \geq 1$ is a feasible control sequence for the problem $\mathcal{P}_1$ at time $k$ and the rest arguments are same as those of the problem $\mathcal{P}_2$. This completes the proof. ∎

In the following theorem, based on Lemma 31 and 32, robust asymptotic stability of the closed-loop system is guaranteed through the monotonic decreasing property of $V(k|k)$ with $P_d(k) \equiv \bar{P}_d$.

**Theorem 1:** (Closed-loop stability) If the optimization problem $\mathcal{P}_2$ is feasible at the initial time $k=0$, then the feasible MPC from $\mathcal{P}_2$ with $P_d(k) \equiv \bar{P}_d$ robustly asymptotically stabilizes the closed-loop system.

**Proof:** To prove the closed-loop stability, we will show that $V(k|k)$ is a strictly decreasing Lyapunov function under A1. First note that by Lemma 32, $\mathcal{P}_1$ and $\mathcal{P}_2$ are always feasible when $\mathcal{P}_2$ is feasible at the initial time $k=0$. Let $P^*(k)$ and $P^*(k+1)$ denote the optimal values of $\mathcal{P}_2$ at time $k$ and $k+1$. By Lemma 3.1, this implies that $P^*(k)$ and $P^*(k+1)$ are also optimal for $\mathcal{P}_1$ at time $k$ and $k+1$, respectively. Thus we have the following inequality from optimality

$$\begin{aligned} &||x(k+1|k+1)||_{P^*(k+1)} + \sum_{i=1}^{d} ||x(k+1-i|k+1)||_{\bar{P}_d} \\ &\quad \leq \ ||x(k+1|k+1)||_{P^*(k)} + \sum_{i=1}^{d} ||x(k+1-i|k+1)||_{\bar{P}_d} \end{aligned} \tag{14}$$

since $P^*(k+1)$ is optimal while $P^*(k)$ is only feasible at time $k+1$. And from (8), we have

$$\begin{aligned} &\left[ ||x(k+1|k)||_{P^*(k)} + \sum_{i=1}^{d} ||x(k+1-i|k)||_{\bar{P}_d} \right] \\ &\quad < \left[ ||x(k|k)||_{P^*(k)} + \sum_{i=1}^{d} ||x(k-i|k)||_{\bar{P}_d} \right] \end{aligned} \tag{15}$$

for any $[A(k)\ \bar{A}(k)\ B(k)] \in \Omega$. Since $x(k+1|k+1)$ equals $[A(k)+B(k)K(k)]x(k|k) + \bar{A}(k)x(k-d|k)$ for some $[A(k)\ \bar{A}(k)\ B(k)] \in \Omega$, and $x(k+1-i|k+1) = x(k+1-i|k), i = 1, ..., d$, they must also satisfy (15). Combining this with (14), we have

$$\begin{aligned} &\left[ ||x(k+1|k+1)||_{P^*(k+1)} + \sum_{i=1}^{d} ||x(k+1-i|k+1)||_{\bar{P}_d} \right] \\ &\quad < \left[ ||x(k|k)||_{P^*(k)} + \sum_{i=1}^{d} ||x(k-i|k)||_{\bar{P}_d} \right] \end{aligned} \tag{16}$$

Thus $V(k|k)$ with $P_d(k) \equiv \bar{P}_d$, i.e., $x^T(k|k)P^*(k)x(k|k) + \sum_{i=1}^{d} x^T(k-i|k)\bar{P}_d x(k-i|k)$, is a strictly decreasing Lyapunov function for the closed-loop system. Since it is bounded below zero, we conclude that $x(k) = x(k|k)$ goes to zero as $k$ goes to infinity. ∎

Now, we summarize the proposed stabilizing MPC algorithm as follows.

**Constrained MPC algorithm for uncertain delayed systems (CMPC-UDS)**
*(1)* (Initialization) at time $k=0$, find $P_d(0)$ by solving the optimization problem $\mathcal{P}_2$, and set $\bar{P}_d \Leftarrow P_d(0)$.
*(2)* (Generic) at time $k \geq 0$, find $K(k)$ by solving the optimization problem $\mathcal{P}_2$ with $P_d(k) \equiv \bar{P}_d$. $P_d(k)$ is not an optimization variable here.
*(3)* Apply the state-feedback control $u(k) = K(k)x(k)$ to the system.
*(4)* At the next time, repeat (2)-(3).

**Remark 1:** One way to choose $\bar{P}_d$ is to solve the problem $\mathcal{P}_2$ temporarily without A1 at the initial time $k=0$, and to set $\bar{P}_d \equiv P_d(0)$. ∎

### 3.2. Modified MPC algorithm

In the previous section, It is shown that the feasible model predictive control robustly asymptotically stabilizes the closed-loop system through the monotonic decreasing property of $V(k|k)$ with $P_d(k) \equiv \bar{P}_d$ for all $k \geq 0$. However, it is somewhat conservative to use a constant matrix $\bar{P}_d$ for all times. In this section, the MPC algorithm will be modified so that $\bar{P}_d$ is updated when certain condition is satisfied.

The following lemma plays an important role in guaranteeing the stability of the closed-loop system with MPC from the modified MPC algorithm.

Lemma 33: Let $P^*(k)$ and $P^*(k+1)$ denote the optimal values of $P(k)$ and $P(k+1)$ for $\mathcal{P}_2$ with $P_d(k+1) \equiv \bar{P}_d$ at time $k$ and $k+1$, respectively. Also let $P_d^*(k+1)$ denotes the optimal value of $P_d(k)$ for $\mathcal{P}_2$ with $P(k+1) \equiv P^*(k+1)$ at time $k+1$. If the following inequality is satisfied

$$P_d^*(k+1) \leq \bar{P}_d \tag{17}$$

then we have

$$\begin{aligned}
&||x(k|k)||_{P^*(k)} + \sum_{i=1}^{d} ||x(k-i|k)||_{\bar{P}_d} \\
&> ||x(k+1|k+1)||_{P^*(k+1)} + \sum_{i=1}^{d} ||x(k+1-i|k+1)||_{\bar{P}_d} \\
&\geq ||x(k+1|k+1)||_{P^*(k+1)} + \sum_{i=1}^{d} ||x(k+1-i|k+1)||_{P_d^*(k+1)}.
\end{aligned} \tag{18}$$

**Proof:** By Lemma 3.1, $P^*(k)$ and $P^*(k+1)$ are also the optimal values of $P(k)$ and $P(k+1)$ for $\mathcal{P}_1$ with $P_d(k+1) \equiv \bar{P}_d$ at time $k$ and $k+1$, respectively. Hence (18) follows from (16) and (17). ∎

**Remark 2:** Note that we cannot derive the condition

$$\begin{aligned}
&||x(k+1|k+1)||_{P^*(k+1)} + \sum_{i=1}^{d} ||x(k+1-i|k+1)||_{\bar{P}_d} \\
&\geq ||x(k+1|k+1)||_{P^*(k+1)} + \sum_{i=1}^{d} ||x(k+1-i|k+1)||_{P_d^*(k+1)}
\end{aligned} \tag{19}$$

directly from the optimality since it does not hold here. Thus we need (17). ∎

Now, based on Lemma 33, we propose a modified MPC algorithm.

**Modified constrained MPC algorithm for uncertain delayed systems (MCMPC-UDS)**

*(1)* (Initialization) at time $k = 0$, find $P_d(0)$ by solving the optimization problem $\mathcal{P}_2$, and set $\bar{P}_d \Leftarrow P_d(0)$ and $flag \Leftarrow 0$.

*(2)* (Generic) at time $k \geq 0$, find $K(k)$ and $P(k)$ by solving the optimization problem $\mathcal{P}_2$ with $P_d(k) \equiv \bar{P}_d$ (thus $P_d(k)$ is not an optimization variable at this step), and set $K^*(k) \Leftarrow K(k)$ and $P^*(k) \Leftarrow P(k)$.

*(3)* Find $K(k)$ and $P_d(k)$ by solving the optimization problem $\mathcal{P}_2$ with $P(k) \equiv P^*(k)$ (thus $P(k)$ is not an optimization variable at this step). If $P_d(k) \leq \bar{P}_d$, set $K^*(k) \Leftarrow K(k)$, $\bar{P}_d \Leftarrow P_d(k)$, $flag \Leftarrow flag + 1$. Otherwise, go to step (5).

*(4)* If $flag \leq r_n$, go to step (2). Otherwise, go to step (5). Here $r_n$ is a given fixed integer that represents the maximum repetition number of (2) and (3).

*(5)* Apply the state-feedback control $u(k) = K^*(k)x(k)$ to the system.

*(6)* At the next time, set $flag \Leftarrow 0$ and repeat (2)-(5).

**Theorem 2:** If the optimization problem $\mathcal{P}_2$ is feasible at the initial time $k = 0$, then the feasible MPC from MCMPC-UDS robustly asymptotically stabilizes the closed-loop system.

**Proof:** Step (3) of MCMPC-UDS exists for the update of $\bar{P}_d$. When $\bar{P}_d$ is not updated in step (3), by Theorem 1, $V(k|k)$ with $\bar{P}_d$ decreases monotonically. When $P_d(k) \leq \bar{P}_d$ and hence $\bar{P}_d$ is updated in step (3), the monotonic decreasing property of $V(k|k)$ with $\bar{P}_d$ still holds by Lemma 33. Since it is bounded below zero, we conclude that $x(k) = x(k|k)$ goes to zero as $k$ goes to infinity. ∎

In the following, the optimization problem $\mathcal{P}_2$ is converted to an optimization problem involving linear matrix inequalities for the implementation of the proposed MPC.

Lemma 34: The optimization problem $\mathcal{P}_2$ can be solved by the following optimization problem

$$\min_{\gamma(k),X(k),Y(k),Z(k),Pd(k)} \gamma(k) \tag{20}$$

subject to

$$0 \leq \begin{bmatrix} 1 & (*) & (*) & \cdots & (*) \\ x(k|k) & X(k) & 0 & \cdots & 0 \\ x(k-1|k) & 0 & X_d(k) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x(k-d^*|k) & 0 & 0 & \cdots & X_d(k) \end{bmatrix}, \tag{21}$$

$$0 \leq \begin{bmatrix} X(k) & (*) & (*) & (*) & (*) & (*) \\ 0 & X_d(k) & (*) & (*) & (*) & (*) \\ M_{31} & \bar{A}_\sigma X_d(k) & X(k) & (*) & (*) & (*) \\ X(k) & 0 & 0 & X_d(k) & (*) & (*) \\ Q^{1/2}X(k) & 0 & 0 & 0 & \gamma I & (*) \\ R^{1/2}Y(k) & 0 & 0 & 0 & 0 & \gamma I \end{bmatrix}, \tag{22}$$

$$0 \leq \begin{bmatrix} Z(k) & Y(k) \\ Y^T(k) & X(k) \end{bmatrix}, Z_{ii}(k) \leq \bar{u}_i^2, \quad i = 1,2,...,m, \tag{23}$$

where $M_{31} = A_\sigma X(k) + B_\sigma Y(k)$, $\sigma = 1,...,p,$, $0 < X(k) = \gamma(k)P^{-1}(k)$, $0 < X_d(k) = \gamma(k)P_d^{-1}(k)$ and $Y(k) = K(k)X(k)$.

**Proof:** See Appendix A.

**Remark 3:** From the modified MPC algorithm MCMPC-UDS, we can see that if the two conditions of steps (3) and (4), $P_d(k) \leq \bar{P}_d$ and $flag \leq r_n$ hold, the steps (2) and (3) are repeated maximally to $r_n$ times at each time $k$. Hence, the size of $r_n$ trade off the computational burdens and the performance. ∎

**Remark 4:** In the optimization problem of Lemma 3..4. when $P(k)$ is not an LMI variable (as in the step (3) of MCMPC-UDS), the $X(k)$ is replaced by $\gamma(k)P^{-1}(k)$. And when $P_d(k)$ is not an LMI variable (as in the step (2) of MCMPC-UDS), the $X_d(k)$ is replaced by $\gamma(k)P_d^{-1}(k)$. ∎

## 4. Numerical example

Consider the uncertain time-varying system with input saturation and time-delay

$$\begin{aligned}
&x(k+1) = A(k)x(k) + \bar{A}(k)x(k-d) + B(k)u(k), \\
&[A(k)\ \bar{A}(k)\ B(k)] \in \text{Co}\{[A_1\ \bar{A}_1\ B_1],\ [A_2\ \bar{A}_2\ B_2]\}, \\
&|u(k)| \leq \bar{u} = 5, \quad d = 2,
\end{aligned}$$

whose system matrices are given

$$A_1 = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 0.5 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.7 & 0.8 \\ 0.05 & -0.3 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Simulation parameters are as follows; the initial value of the state is $x(0) = [5\ 1]^T$; the upper bound of $d$ is $d^* = 3$; the state and the input weighting matrices are $Q = \mathrm{diag}(1,1)$ and $R = 1$.

Simulation results are given in Figure 1 where the states converge to zero as time goes to infinity and the input satisfies the input constraints (Figure 2). Figure 3 shows how the the cost vary for $r_n = 1, 10$ and 20, respectively.

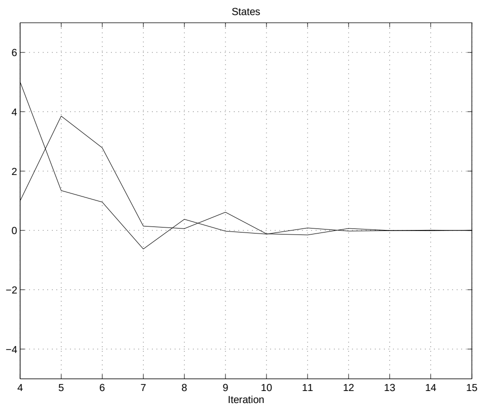


Fig. 1. Trajectories of states

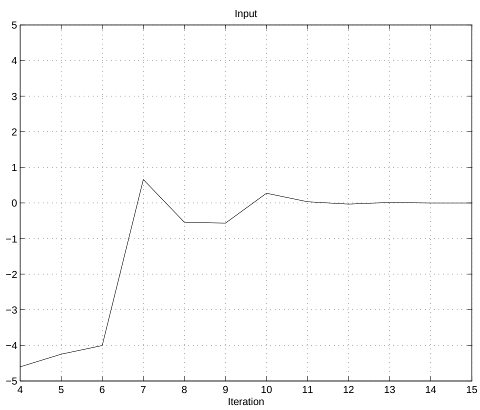


Fig. 2. Trajectories of inputs

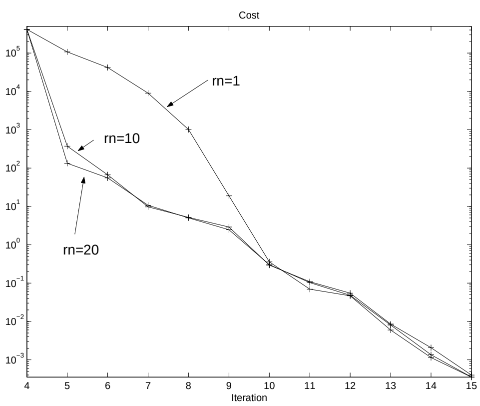


Fig. 3. Comparison of costs

## 5. Concluding Remarks

In this paper, we presented an MPC algorithm for uncertain time-varying systems with input constraints and state-delay. The uncertainty was assumed to be the type of polytopic uncertainty, and the delay was unknown but its upper bound is assumed to be known in advance for practical reason. To find memoryless state-feedback MPC law, an optimization problem which minimizes a cost function at each sampling time was considered. After finding the upper bound of the cost function by assuming certain inequality, the original problem is relaxed to another optimization problem from which we derive the final form of optimization problem involving LMIs. It was shown that a feasible solution to the problem at time $k$ is also a feasible solution to the same problem after the time $k$ under certain conditions. Finally, based on the feasibility and optimality, the closed-loop stability was established. Through an example, we showed the nice performance of the proposed controller.

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### Appendix A

We prove (20), (21) and (23) here. Minimization of $V_u(k|k)$ is equivalent to

$$\text{Minimize } \gamma(k) \tag{24}$$

subject to

$$\gamma(k) \geq ||x(k|k)||_{P(k)} + \sum_{i=1}^{d^*} ||x(k-i|k)||_{P_d(k)}. \tag{25}$$

By defining $X(k) = \gamma(k)P^{-1}(k) > 0$, $X_d(k) = \gamma(k)P_d^{-1}(k) > 0$ and using the Schur complement, (25) is equivalent to

$$0 \leq \begin{bmatrix} 1 & (*) & (*) & \cdots & (*) \\ x(k|k) & X(k) & 0 & \cdots & 0 \\ x(k-1|k) & 0 & X_d(k) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x(k-d^*|k) & 0 & 0 & \cdots & X_d(k) \end{bmatrix}, \tag{26}$$

which establishes (20) and (21).

Next, we relax the input constraint (6) by using the so called invariant ellipsoid. To this ends, let us define an ellipsoid $E_P(k)$ centered at the origin

$$E_P(k) = \{\xi \mid \xi^T P(k)\xi \leq \gamma(k)\}. \tag{27}$$

The following lemma help us to relax the input constraint (6).

Lemma 51: Suppose that $V(k|k) \leq \gamma(k)$, and that there exist a $K(k)$ and a positive definite matrix $P(k)$ satisfying both (8) and the following LMIs

$$\begin{bmatrix} Z(k) & K(k) \\ K^T(k) & \gamma^{-1}(k)P(k) \end{bmatrix} \geq 0, \quad Z_{ii}(k) \leq \bar{u}_i^2, \tag{28}$$

where $i = 1, 2, ..., m$ and $\bar{u}_i$ is the $i$th element of $\bar{u}$. Then the state-feedback controller $u(k+j|k) = K(k)x(k+j|k)$ stabilizes the system for all $x(k|k) \in E_P(k)$ while satisfying the input constraints (2). And the resultant state trajectory $x(k+j|k)$ always remains in the region $E_P(k)$.

**Proof:** Under the assumptions, it is apparent that $x^T(k+j|k)P(k)x(k+j|k) < \gamma(k)$ for all $j \geq 0$. Hence we can rewrite the input constraints (6) as follows (see [17]):

$$\begin{aligned} |u(k+j|k)|^2 =& \ |K(k)x(k+j|k)|^2 \\ &= |K(k)P^{-1/2}(k)P^{1/2}(k)x(k+j|k)|^2 \\ &\leq \ ||K(k)P^{-1/2}(k)||_2^2||P^{1/2}(k)x(k+j|k)||_2^2 \\ &= \ K(k)P^{-1}(k)K^T(k)x^T(k+j|k)P(k)x(k+j|k) \\ &< \ K(k)P^{-1}(k)K^T(k)\gamma(k) \ \leq \ \bar{u}^2, \end{aligned} \tag{29}$$

which is followed by the inequality (28). Therefore, if $V(k|k) \leq \gamma(k)$ and there exist a $K(k)$ and a positive definite matrix $P(k)$ satisfying both (8) and (28), the predicted state remains in $E_P(k)$ for all times while satisfying the input constraints. ■

We can utilize the results of Lemma 5.1 to incorporate the input constraints into optimization problem as sufficient LMI constraints. Pre- and post-multiplying by

$$\begin{bmatrix} I & 0 \\ 0 & \gamma(k)P^{-1}(k) \end{bmatrix} \tag{30}$$

and substituting $X(k) = \gamma(k)P^{-1}(k)$, $X_d(k) = \gamma(k)P_d^{-1}(k) > 0$ and $Y(k) = K(k)X(k)$, we see that (28) is equivalent to (23). This completes the proof.