Error Reduction of Sliding Mode Control Using Sigmoid-Type Nonlinear Interpolation in the Boundary Layer

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Abstract: Sliding mode control with nonlinear interpolation in the boundary layer is proposed. A modified sigmoid function is used for nonlinear interpolation in the boundary layer and its parameter is tuned by a fuzzy logic controller. The fuzzy logic controller that takes the distance between the system state and the sliding surface as its input guides the choice of parameter of the modified sigmoid function and the parameter is on-line tuned. Owing to the decreased thickness, the proposed method has better tracking performance than the conventional linear interpolation method. To demonstrate its performance, the proposed control algorithm is applied to a simple nonlinear system model.

Keywords: sliding mode control, fuzzy logic controller, boundary layer thickness, modified sigmoid function, nonlinear interpolation

1. INTRODUCTION

Sliding mode control (SMC) is a robust nonlinear feedback control technique with the drawback of chattering. The most common method for solving chattering problem is to introduce a boundary layer (BL) around the switching surface and to use continuous control within the BL. This method, however, does not ensure the convergence of the state trajectory of the system to the sliding surface, and probably results in the existence of the steady-state error.

To reduce the steady-state error, a number of studies have been published. In [1], the nonlinear interpolation in BL is used to better cope with changing system dynamics, unknown model uncertainties, and disturbances. Fuzzy logic controllers (FLCs) were employed to adapt the specific parameters which characterize nonlinear interpolation in BL. An S-shaped continuous function in BL is used to avoid chattering. On the other hand, due to the similarity between FLC and SMC, SMC has been used to guarantee the stability and robustness of the FLC systems [2]. In the fuzzy sliding mode controller (FMSC), the fuzzy transfer function (operating line) in BL is composed of S-shaped functions, which has the characteristics of strictly monotonous decreasing and symmetric, to better cope with disturbance and parameter uncertainties. In above papers, S-shaped nonlinear functions are used to interpolate nonlinearly in BL and the S-shape interpolation in BL can give a bigger control input than linear interpolation at the same switching function value. Therefore, system states can converge fast to the sliding surface. At the same time, the tracking error is smaller than linear interpolation because smaller boundary layer allows us to make much use of the available bandwidth. To adapt the transfer function according to a performance criterion, this algorithm need complicated processing that is required to another FLC to change the shape of the input membership functions, however.

In this paper, a modified sigmoid function (MSF), which is one of the S-shaped functions, is proposed to interpolate nonlinearly in the BL and the parameter of the MSF is controlled by a FLC as shown in Fig. 1. The fuzzy rules are introduced to control the shape of the modified sigmoid function and the inputs of the FLC depend on the distance from the switching line. The advantage of using the MSF is that its shape can be controlled simply by a single parameter and the shape of the MSF can be made easily similar to the transfer function of adaptive FMSC, which is required two FLCs, by controlling its parameter.

Computer simulation examples using the proposed method for a simple nonlinear model and a DC motor are executed to show the performance of the proposed algorithm.

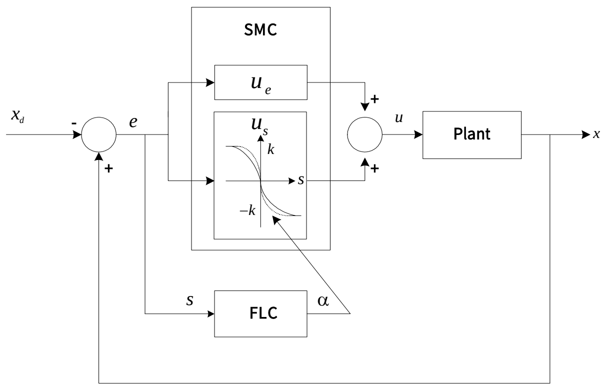


Fig. 1 Configuration of the control system.

2. SLIDING MODE CONTROL

Consider a single-input second order system

$$\ddot{x} = f(\mathbf{x}) + d(t) + u, \tag{1}$$

where the scalar x is the output of interest, $\mathbf{x} = [x\ \dot{x}]^T$ is the state vector, $d(t)$ is the disturbance, $f(\mathbf{x})$ is a unknown function, and the scalar u is the control input. We assumed that the function $f(\mathbf{x})$ is

$$f(\mathbf{x}) = \hat{f}(\mathbf{x}) + \Delta f(\mathbf{x}), \tag{2}$$

where $\hat{f}(\mathbf{x})$ and $\Delta f(\mathbf{x})$ are the estimation of $f(\mathbf{x})$ and the model uncertainty, respectively. The model uncertainty and the disturbance are assumed to be bounded as

$$\left|\Delta f(\mathbf{x})\right| = \left|f(\mathbf{x}) - \hat{f}(\mathbf{x})\right| \le F(\mathbf{x}) \text{ and } \left|d(t)\right| \le D(t). \tag{3}$$

Let the time varying sliding surface s be expressed in the state-space $\mathbf{R}^2$ by $s(\mathbf{e}) = 0$ as

$$s(e) = (\frac{d}{dt} + \lambda)e = \dot{e} + \lambda e, \quad \lambda > 0 \tag{4}$$

and define the tracking error $\mathbf{e} = \mathbf{x} - \mathbf{x}_d = [e \ \dot{e}]^T$ where $\mathbf{x}_d = [x_d \ \dot{x}_d]^T$ is the desired state vector. The control input to get the state $\mathbf{x}$ to track a specific time-varying desired state $\mathbf{x}_d$ in the presence of model uncertainty on $f(\mathbf{x})$ is made to satisfy the following sliding condition [3]:

$$\frac{1}{2}\frac{d}{dx}s^2 \le -\eta |s|, \quad \eta \ge 0. \tag{5}$$

Let $\hat{u}$ be the nominal control law that can be interpreted as the best estimate, computed by $\dot{s} = 0$ with known information $f = \hat{f}$, then it is found as follows:

$$\hat{u} = \ddot{x}_d - \lambda\dot{e} - \hat{f}. \tag{6}$$

Thus, the control law that satisfies the sliding mode condition Eq. (5) can be obtained as

$$u = \hat{u} - u_s$$
$$u_s = k\,\mathrm{sgn}(s)$$
$$\mathrm{sgn}(s) = \begin{cases} +1, & if\ s > 0 \\ -1, & if\ s < 0 \end{cases}, \tag{7}$$

where $k \ge F + \eta$.

A certain disadvantage of this method is the drastic changes of the control input, which leads to high stress for the plant to be controlled. However, this can be avoided by means of BL near the switching line which smoothes out the control behavior and ensures that the system states remain within this layer. Therefore, we substitute the signum function $\mathrm{sgn}(s)$ in Eq. (5) by saturation function $sat(s/\phi)$, where ϕ is the BLT.

$$u = \hat{u} - \bar{k}\,sat(s/\phi) \tag{8}$$

with $sat(x) = \begin{cases} x & if\ |x| < 1 \\ \mathrm{sgn}(x) & otherwise, \end{cases}$ and $\bar{k} = k - \dot{\phi}$.

From Eqs. (1), (3), (6) the filter function follows:[3]

$$\dot{s} = -\beta s + (-\Delta f(\mathbf{x_d}) + O(\varepsilon)),\ \beta \triangleq \frac{\bar{k}}{\phi} \tag{9}$$

and $O(\varepsilon)$ represents a term of relatively small magnitude caused by using a desired state instead of actual state vector in Eq. (9). This filter with bandwidth β removes the high-frequency chattering to give a smooth s. Fig. 2 shows the first-order low-pass filter (LPF) for s, where p is the Laplace operator $\frac{d}{dt}$.

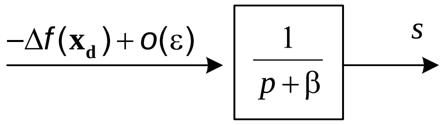


Fig. 2 Filter for chattering reduction.

3. FUZZY SLIDING MODE CONTROL

A FLC which uses the switching variable s as the input to calculate the control variable u belongs to the family of FSMCs. In [1], the FLC is used to interpolate in BL. Due to the strictly monotonous decreasing u as s increases in the BL, the shape of the nonlinear transfer characteristic(NTS) of the FSMC becomes S-shape depends not only on the values of u, but also on the membership functions of the rule antecedents and consequents and the defuzzification method. Similar to the SMC with BL, the rules are, in general, such conditioned that above the switching line a negative control output is generated and a positive one below it. The pattern of the control can be expressed by the following several rules:

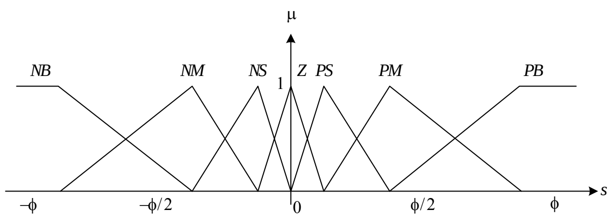
Rule R^1: If s is NB then u is PB
Rule R^2: If s is NM then u is PM
Rule R^3: If s is NS then u is PS
Rule R^4: If s is Z then u is Z
Rule R^5: If s is PS then u is NS
Rule R^6: If s is PM then u is NM
Rule R^7: If s is PB then u is NB

where NB; Negative Big, NM; Negative Medium, NS; Negative Small, Z; Zero, PB; Positive Big, PM; Positive Medium, PS; Positive Small.

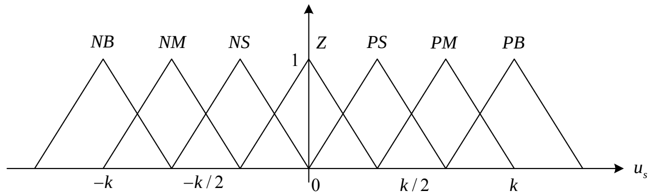
Due to similarity between FLCs and SMCs [2], we consider the following fuzzy sliding control law:

$$u(t) = \hat{u}(t) - u_s$$
$$u_s = -k_{fuzzy}(e, \dot{e}, \lambda) \cdot sat(s/\phi), \tag{10}$$

where $k_{fuzzy}(e, \dot{e}, \lambda)$ is the absolute value of the control output of the FLC. The nonlinear transfer characteristics of the above fuzzy rules, of which input and output membership are shown in Fig. 3, have a reverse S-shaped configuration as drawn in Fig. 4. The graph represents the control value u_s against s after the center-of-sums defuzzification.



(a)



(b)

Fig. 3 Membership functions for the fuzzy sets. (a) Input s. (b) Output u_s.

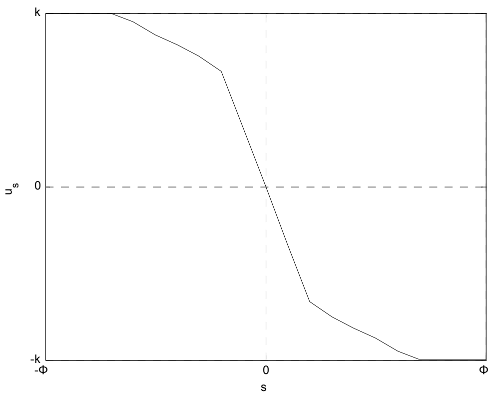


Fig. 4 Nonlinear transfer characteristics of the FSMC.

4. NONLINEAR INTERPOLATION USING A SIGMOID FUNCTION

In order to eliminate the chattering caused by the signum function in Eq. (7) we introduce a MSF, which has the similar shape to the NTSs of the FMSC, to replace the signum function when the states of the system get into the BL. The MSF used in this paper is

$$f(x;\alpha) = -\frac{2}{1+\mathrm{e}^{-\alpha x}} + 1, \tag{11}$$

where α is the constant which controls the shape of the function. The MSF and the varying boundary layer(VBL), which will be defined later, are shown in Fig. 5.

The signum function in Eq. (7) to avoid abrupt change of the input. Thus, the new control law within the BL is expressed as follows;

$$u(t) = \hat{u}(t) - u_s$$

$$u_s = -k\frac{2}{1+\mathrm{e}^{-\alpha s}} + 1. \tag{12}$$

We define the virtual boundary layer (VBL) as twice of the distance from $s = 0$ to the maximum s value ($s = \phi_1$ in Fig. 5) at the point that the MSF can be linearly approximated. The graphical view of VBL is shown in Fig. 5, where $-k_1$ is the u_s value at ϕ_1 and ϕ is the BLT.

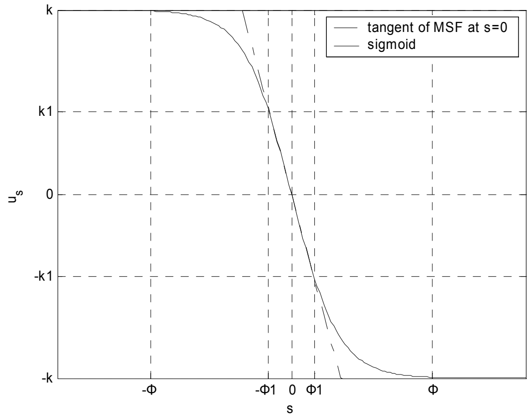


Fig. 5 Definition of the VBL.

In the nonlinear interpolation using MSF, we can regarded the BLT as ϕ_1 instead of ϕ in Fig. 5 if $s \le |\phi|$. Thus, the steady state error is better than the linear interpolation since the narrowed the BLT allows us to better use of the bandwidth in Eq. (9)

From Eqs. (4), (5), (9), and Fig. 2, the structure of the closed-loop dynamics can be summarized in Fig. 6 [3].

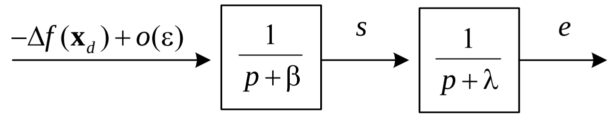


Fig. 6 Structure of the closed-loop dynamics.

From Fig. 6, the 2nd-order filter function can be given in Eq. (13)

$$e = \frac{1}{p+\lambda}\cdot\frac{\Delta f(\mathbf{x}_\mathrm{d}) + O(\varepsilon)}{p+\beta} \tag{13}$$

To give a critical damping of the 2nd-order filter function, we let $\beta = \lambda$ and rewrite Eq. (13) as

$$e = \frac{\Delta f(\mathbf{x}_\mathrm{d}) + O(\varepsilon)}{(p+\lambda)^2} \tag{14}$$

If $s \le |\phi_1|$, we can replace a pole $-\beta$ in Eq. (13) to $-\beta_1$ giving a wider bandwidth which results in better error reduction by forcing the pole to the more negative direction on real axis. Thus, we can reduce the effect of uncertainty $\Delta f(\mathbf{x}_d)$ without chattering.

From Eqs. (8),(9), and Fig. 5,

$$\beta = \frac{k-\dot{\phi}}{\phi} < \beta_1 = \frac{k_1-\dot{\phi}_1}{\phi_1} \tag{15}$$

The nonlinear transfer characteristics in BL of this type of SMC depend on the MSF parameter "α". A FLC is employed to calculates the parameter "α" in Eq. (11) in order to control the shape of the MSF. To estimate the range of "α", a measure of chattering, Γ is introduced [5].

$$\Gamma = |\dot{u}|. \tag{16}$$

The range of parameter " α " is

$$\alpha_0 < \alpha < \alpha_\Gamma, \tag{17}$$

where α_0 is the value of the α at which the BLT of linear interpolation and α_Γ is the maximum parameter before chattering occurs.

The main idea of the developed FLC can be the control law: *"If the switching function s is small, increase the parameter " α " to increase the control (and conversely)"*. Therefore, the following properties on the FLC can be inferred qualitatively depending on s :

$$\alpha = \begin{cases} big & s \cong 0 \\ small & |s| > 0 \end{cases}. \tag{18}$$

The pattern of the control rules of Eq. (18) can be expressed by following three rules when we define $\alpha \triangleq \alpha_0 + \Delta\alpha$.

IF s is almost zero THEN $\Delta\alpha$ is PB
IF s is small THEN $\Delta\alpha$ is PS
IF s is big THEN $\Delta\alpha$ is ZR

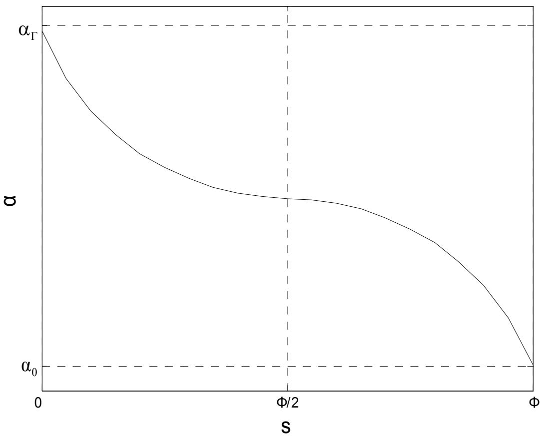


Fig. 7. Output " α " of the FLC.

Consequently, we have the following design procedure.

Step 1) Calculate the parameter " α_0 " at given BLT of linear interpolation.
Step 2) Obtain the parameter " α " from the FLC.
Step 3) Calculate the u in Eq. (12).
Repeat steps 2) and 3).

The u_s in Eq. (12) only converges to $-k$ if s goes to infinity because of the characteristics of MSF. There exists a discontinuity between k and k' at the fixed BL ϕ in Fig. 8 and the discontinuity may causes chattering. This problem can be solved by compensating the gap, $\left|k - k'\right|$ in Fig. 8. Now, we can eliminate a chattering by multiplying u_s by $\frac{k}{k'}$ as

$$u_s = k\left(\frac{-2}{1 + e^{-\alpha s}} + 1\right)\frac{k}{k'} \tag{19}$$

Because the amount of compensation is very small, the effect of changing for shape of the MSF can be ignored.

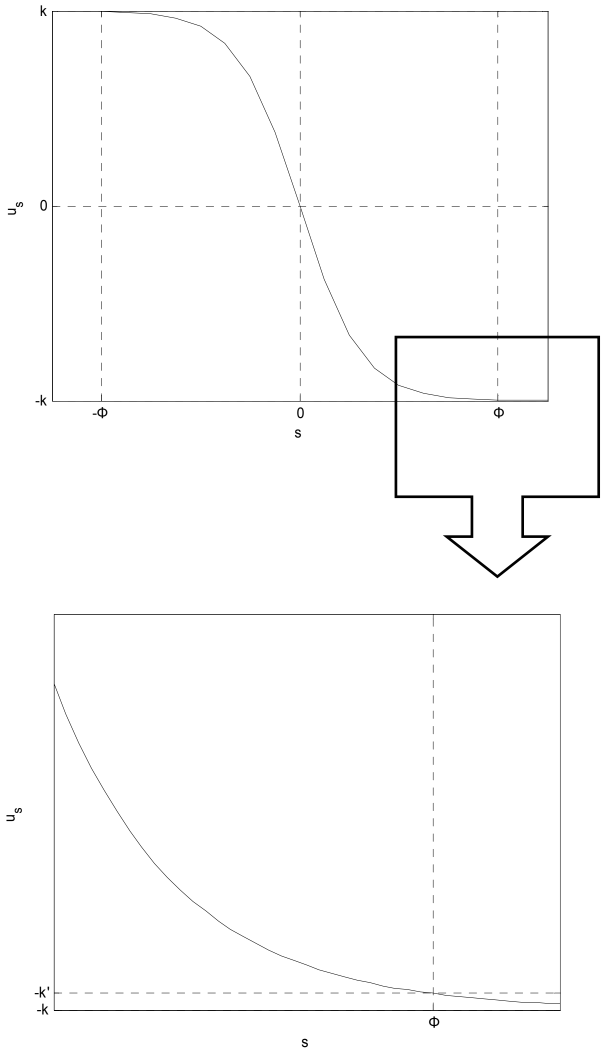


Fig. 8. The problem of discontinuity around the BLT.

5. STABILITY

To design a stable SMC using MSF in BL, we must determine proper values of $\hat{u}$ and k in Eq. (12). First, the approximation of input, $\hat{u}$ would be obtained from Eq. (6). Next, we can choose the value of k by the bounds on f and d. Now the following theorem gives the coordinate system ultimately bounded ness [2].

Theorem: For the system (1), the proposed SMC which has the MSF in BL makes the system trajectories ultimately bounded under the region $B = \{|s| \le \phi\}$ *with* $\phi > 0,$ if we choose k as

$$k \ge F + D + \eta, \tag{20}$$

where η is a strictly positive constant

Proof. Define the Lyapunov function

$$V = \frac{1}{2}s^2. \quad (21)$$

Then the derivative $\dot{V}$ along the system trajectory is

$$\begin{aligned}\dot{V} &= s\cdot\dot{s} \\ &= s\cdot(\lambda\dot{e} + f(x,t) + d(t) - \ddot{x}_d + u) \\ &= s\cdot(f - \hat{f} + d(t) - k\,sat(\frac{-2}{1+e^{-\alpha s}}+1)\,\frac{k}{k'}). \end{aligned} \quad (22)$$

$if\ |s| > \phi,$

$$\dot{V} \le s\cdot(F + D - k\,sat(\frac{-2}{1+e^{-\alpha s}}+1)\,\frac{k}{k'}) = s\cdot(F+D) - k|s|$$

$$\le -\eta|s|$$

Therefore, using the value k of Eq. (20), we always have the bounded system trajectories under the range of B.

6. ILLUSTRATIVE EXAMPLE

A simplified nonlinear model of the motion of an underwater vehicle can be written [3]

$$\begin{aligned} &m\ddot{x} + c\dot{x}|\dot{x}| = u \\ &\ddot{x} = -\frac{c\dot{x}|\dot{x}|}{m} + \frac{u}{m};\ 1 \le m \le 5;\ 0.5 \le c \le 1.5 \\ &f = -\frac{c\dot{x}|\dot{x}|}{m};\ m = 3 + 1.5\sin(|\dot{x}|t) \\ &c = 1.2 + 0.2\sin(|\dot{x}|t) \\ &\hat{f} = -\frac{\dot{x}|\dot{x}|}{\sqrt{5}};\ \hat{m} = \sqrt{5};\ \hat{c} = 1 \\ &F = 0.3\dot{x}|\dot{x}|;\ \eta = 0.1;\ \lambda = 10 \\ &x_d = \sin(\pi t/2). \end{aligned} \quad (23)$$

where x defines position, u is the control input, m is the mass of the vehicle, c is a drag coefficient, and the sampling rate is 0.002s. The boundary ϕ is made varying and the control law Eq. (8) is used which for this example is

$$u = c\dot{x}|\dot{x}| + \ddot{x}_d - 10\dot{e} - \bar{k}\cdot \mathrm{sgn}(s/\phi) \quad (24)$$

with

$$\begin{aligned} &k = 0.3\dot{x}|\dot{x}| + 0.1\sqrt{5} + \sqrt{5}(\sqrt{5}-1)\,|\,\ddot{x}_d - 10\dot{e}\,| \\ &k(x_d) \ge \frac{10\phi}{\sqrt{5}} \ \Rightarrow\ \dot{\phi} + 10\phi = \sqrt{5}\,k(x_d) \quad \bar{k} = k - \dot{\phi}/\beta \\ &k(x_d) \le \frac{10\phi}{\sqrt{5}} \ \Rightarrow\ \dot{\phi} + \frac{10\phi}{5} = k(x_d)/\sqrt{5} \quad \bar{k} = k - \dot{\phi}\beta. \end{aligned}$$

Now, the MSF is used to interpolate in VBL. Control law Eq. (19) is used as

$$u = c\dot{x}|\dot{x}| + \ddot{x}_d - 10\dot{e} - \bar{k}\,(\frac{-2}{1+\mathrm{e}^{-\alpha s}}+1)\,\frac{k}{k'} \quad (25)$$

where "α" is the output of FLC.

The corresponding control input, s-trajectories and tracking error are plotted in Figs. 10 ~ 12 when the desired trajectory are given as Fig. 9.

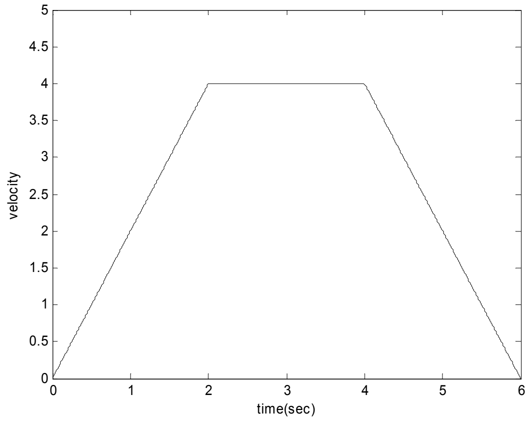


Fig. 9. Desired trajectory.

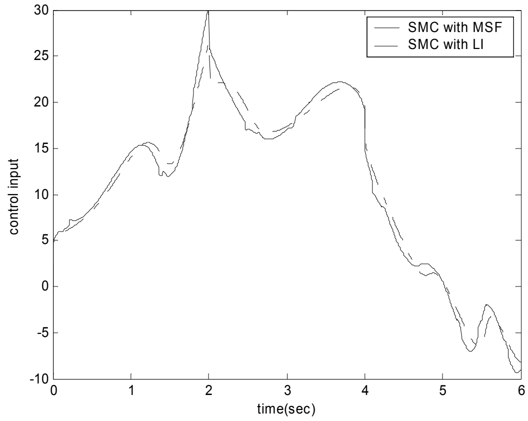


Fig. 10. Control input.

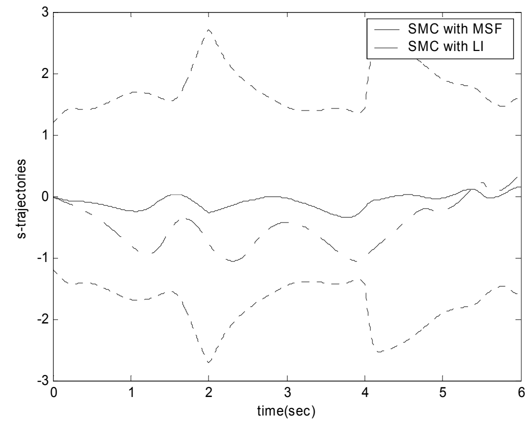


Fig. 11. s-trajectories with time-varying BLs.

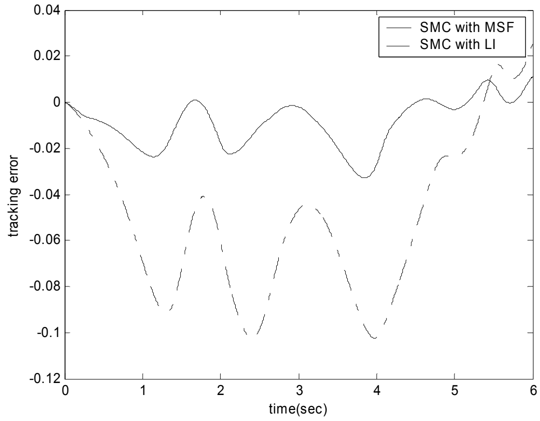


Fig. 12. Tracking performance.

7. DISCUSSIONS

The MSF has been employed for nonlinear interpolation in BL, as opposed to the conventional sliding mode controller with linear interpolation in the fixed BL or the traditional variable BL with the linear interpolation. After the operating range of a parameter which used to control the shape of MSF is on-lined updated, the parameter is tuned by the FLC.

Due to a wider bandwidth which give a fast response of the error filter function, we can reduce the steady state error can be reduced under the system uncertainties and disturbances. By computer simulations, the proposed controller has shown to produce the smaller steady-state error than the conventional one.

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