Tests for Seasonal Cointegrating Vectors

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Abstract

We obtain the asymptotic distributions of tests statistics for various types of seasonal cointegration based on GRR estimators of Ahn and Cho (2003). These tests are useful in testing for restrictions about cointegrating vectors after Chi-square tests for CCI and common PCIV in Ahn and Cho (2003) or tests for the known CCI and the known PCIVs have been performed.

Key Words: Seasonal cointegration, Gaussian reduced rank estimator, Contemporaneous cointegration, Polynomial cointegrating vector

1. Introduction

Unlike the approaches of Johansen and Schaumbug (1999) and Cubadda (2001), GRR estimation in Ahn and Reisel (1994) and Ahn and Cho (2003) enables estimation of models with common PCIV's at different seasonal unit roots imposed. In this thesis, we obtain the asymptotic distributions of tests statistics for various types of seasonal cointegration based on GRR estimators of Ahn and Cho (2003). These can be easily deduced using the results in Ahn and Reinsel (1994) and Ahn and Cho (2003). These tests are useful in testing for restrictions about cointegrating vectors after Chi-square tests for CCI and common PCIV in Ahn and Cho (2003) or tests for the known CCI and the known PCIV's have been performed.

2. Model and Error Correction Representation

Let y_t be an m-dimensional autoregressive (AR) process with nonstationary seasonal behavior with period s such that

$$\Phi(L)y_t = (I_m - \sum_{k=1}^{p} \Phi_k L^k) y_t = \varepsilon_t , \tag{1}$$

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where L is a lag operator such that $L\mathbf{y}_t = \mathbf{y}_{t-1}$, and $\boldsymbol{\varepsilon}_t$ are assumed to be independent with $E(\boldsymbol{\varepsilon}_t) = 0$, $Cov(\boldsymbol{\varepsilon}_t) = \Omega$. We assume that each component of $\mathbf{y}_t$ with a first-order seasonal differencing of period s exhibits stationary behavior. That is, $\mathbf{z}_t = (1-L^s)\mathbf{y}_t = \mathbf{y}_t - \mathbf{y}_{t-s}$ stationary. Let us further assume that $\det\{\Phi(L)\} = 0$ has some roots on the unit circle and the remaining roots are outside the unit circle.

For a nonstationary distinct root $\omega_j = \exp(i\theta_j)$ for $\theta_j \in [0,\pi]$ and $j = 1,\ldots,K$, where $i = \sqrt{-1}$ is an imaginary number, we assume that $rank\{\Phi(\omega_j)\} = r_j$, $r_j = m - d_j > 0$ and define, as in Cubadda (2001),

$$\nabla_j(L) = \begin{cases} 1-\cos\theta_j L & for\ \ \theta_j = 0\ or\ \pi \\ 1-2\cos\theta_j L + L^2 & for\ \ \theta_j \in (0,\pi) \end{cases},$$

and

$$f^{(j)}(L) = \prod_{k \neq j}^{K} \nabla_k(L)$$

which can be viewed as a linear filter to remove nonstationary roots except ω_j and its conjugate $\overline{\omega}_j$ or as a Lagrange polynomial from approximation theory. In additon it is assumed that

$$\begin{cases} -\Phi(\omega_j)/f^{(j)}(\omega_j) = \alpha_j\beta_j' & for\ \ \theta_j = 0\ or\ \pi \\ \Phi(\omega_j)/\{i\omega_j f^{(j)}(\omega_j)\sin\theta_j\} = \alpha_j\beta_j' & for\ \ \theta_j \in (0,\pi) \end{cases} \tag{2}$$

is a full rank factorization of the scaled $\Phi(\omega_j)$ where $\alpha_j = \alpha_{jR} + \alpha_{jI}i$ and $\beta_j = \beta_{jR} + \beta_{jI}i = [I_{r_j}, \beta_{0jR}']' + [O_{r_j}, \beta_{0jI}']'i$ are complex-valued $m \times r_j$ matrices of full rank, α_{jR}, α_{jI}, β_{jR} and β_{jI} are $m \times r_j$ real-valued matrices.

Following Ahn and Cho (2003), the error correction representation for model (1) can be written as

$$\Phi^*(L)(1-L^s)\mathbf{y}_t = \sum_{\theta_j \in 0\, or\, \pi} \alpha_{jR}\beta_{jR}'\mathbf{y}_{t-1}^{(j)}$$

$$+ \sum_{\theta_j \in (0,\pi)} \{(\alpha_{jR}\beta_{jI}' + \alpha_{jI}\beta_{jR}')\mathbf{w}_{t-1}^{(j)} + (\alpha_{jI}\beta_{jI}' - \alpha_{jR}\beta_{jR}')\mathbf{v}_{t-1}^{(j)}\} + \boldsymbol{\varepsilon}_t \tag{3}$$

where $\mathbf{y}_{t-1}^{(j)} = f^{(j)}(L)\mathbf{y}_{t-1}$ is the filtered series of $\mathbf{y}_{t-1}$ with all the nonstationary roots except ω_j and $\overline{\omega}_j$ removed, $\mathbf{w}_t^{(j)} = \sin\theta_j \mathbf{y}_t^{(j)}$, and

$$\mathbf{v}_t^{(j)} = \begin{cases} -\cos\theta_j \mathbf{y}_t^{(j)} & for\ \ \theta_j = 0\ or\ \pi \\ -\cos\theta_j \mathbf{y}_t^{(j)} + \mathbf{y}_{t-1}^{(j)} & for\ \ \theta_j \in (0,\pi) \end{cases}.$$

We define $\boldsymbol{\beta}_j = (\boldsymbol{\beta}_{jR}', \boldsymbol{\beta}_{jI}')'$ with $\boldsymbol{\beta}_{jR} = vec(\beta_{0jR})$, $\boldsymbol{\beta}_{jI} = vec(\beta_{0jI})$, $\boldsymbol{\alpha} = vec\{(\leftarrow \alpha_{jR}, \alpha_{jI} \rightarrow, \Phi_1^*, \ldots, \Phi_{p-s}^*)'\}$, and $\boldsymbol{\eta} = (\leftarrow \boldsymbol{\beta}_{jR}', \boldsymbol{\beta}_{jI}' \rightarrow, \boldsymbol{\alpha}')'$ where $vec(A)$ denotes a vector formed by stacking the columns of matrix A and $\leftarrow \alpha_{jR}, \alpha_{jI} \rightarrow$ denotes an arrangement of α_{jR}, α_{jI} side by side for all j's.

3. Tests for Cointegrating Vectors in the Presence of Contemporanous Cointegration or Common Polynomial Cointegrating Vectors

It is often of interest to test if a seasonally cointegrated time series is contemporaneously cointegrated or if it has common PCIV's corresponding to different seasonal unit roots. Contemporanous seasonal cointegration or existence of common PCIV's lead to not only parsimonious modeling but also different or new interpretation of the process (Lee, 1992; Cubadda, 2001). Ahn and Cho (2003) developed tests for contemporaneous cointegration and for common PCIV's at different frequencies. They use the Chi-squared test statistic based upon the GRR estimator for testing these types of cointegration.

In thie thesis we study further tests for simple linear restrictions on seasonal cointegrating vectors such as (i) $H_0 : \boldsymbol{\beta}_{jR} = \mathbf{b}$ in the presence of contemporaneous cointegration, (ii) $H_0 : \boldsymbol{\beta}_{jR} = \boldsymbol{\beta}_{kR} = \mathbf{b}$ for $j \neq k$ in the presence of common contemporaneous cointegration at different frequencies, and (iii) $H_0 : (\boldsymbol{\beta}'_{jR}, \boldsymbol{\beta}'_{jI})' = (\boldsymbol{\beta}'_{kR}, \boldsymbol{\beta}'_{kI})' = (\mathbf{b}'_R, \mathbf{b}'_I)'$ for $j \neq k$ in the presence of common PCIV's at different frequencies. These tests are useful for the further analysis after the researchers test for contemporaneous cointegration or common PCIV's using Theorem 2 of Ahn and Cho (2003). Or these may be useful for testing the known contemporaneous cointegration or the known common PCIV's.

First, we consider the case: $H_0 : \boldsymbol{\beta}_{jR} = \mathbf{b}$ in the presence of contemporaneous cointegration. Contemporaneous cointegration corresponding to the seasonal unit root ω_j means $\boldsymbol{\beta}_{jI} = 0$ (Ahn and Reinsel, 1994). Theorem 1 shows the asymptotic distribution of this case and its asymptotic normality enables us to test for $H_0 : \boldsymbol{\beta}_{jR} = \mathbf{b}$ in the presence of contemporaneous cointegration using the Chi-squared distribution.

Theorem 1. In the model with no deterministic term, let $\hat{\boldsymbol{\eta}}$ *denote the Gaussian estimator for* $\boldsymbol{\eta} = (\leftarrow \boldsymbol{\beta}'_{jR,} \boldsymbol{\beta}'_{jI} \rightarrow, \boldsymbol{\alpha}')'$. *In the presence of contemporaneous cointegration the asymptotic distribution of* $\hat{\boldsymbol{\beta}}_{jR}$ *is, when the correct model with* $\boldsymbol{\beta}_{jI} = 0$ *is fitted,*

$$T\left(\hat{\boldsymbol{\beta}}_{jR} - \mathbf{b}\right) \rightarrow^{D} \left(\tilde{F}_{j33}\right)^{-1} vec(\tilde{G}_{j3}\Omega^{-1}\alpha_{jI} - \tilde{G}_{j4}\Omega^{-1}\alpha_{jR}), \tag{4}$$

for $\theta_j \in (0, \pi)$.

In addition,

$$\left(\sum \hat{X}_{t-1}^{(j)} \hat{\Omega}^{-1} \hat{X}_{t-1}^{\prime(j)}\right)^{1/2} \left(\hat{\boldsymbol{\beta}}_{jR} - \mathbf{b}\right) \rightarrow^{D} N(\mathbf{0}, I_{r_j d_j})$$

where $\hat{X}_{t-1}^{\prime(j)} = [\hat{\alpha}_{jI} \otimes \mathbf{w}_{2t-1}^{\prime(j)} - \hat{\alpha}_{jR} \otimes \mathbf{v}_{2t-1}^{\prime(j)}]$ *is a* $m \times r_j d_j$ *matrix,* $\hat{\Omega}$ *is a consistent estimator of* Ω.

Next, we consider testing for the hypothesis $H_0 : \boldsymbol{\beta}_{jR} = \mathbf{b}_{jR}$ in the presence of common contemporaneous cointegration at different frequencies. Common contemporaneous cointegration corresponding to different seasonal unit roots means $\boldsymbol{\beta}_{jR} = \boldsymbol{\beta}_{kR}$ and $\boldsymbol{\beta}_{jI} = \boldsymbol{\beta}_{kI} = 0$ for $j \neq k$.

Theorem 2. Under the assumptions of Theorem 1, the asymptotic distribution of $\hat{\boldsymbol{\beta}}_{jR}$ *in the presence of common contemporaneous cointegration is, when the correct model with* $\boldsymbol{\beta}_{jR} = \boldsymbol{\beta}_{kR}$ *for* $j \neq k$ *is fitted,*

$$T\left(\hat{\boldsymbol{\beta}}_{jR} - \mathbf{b}\right) \rightarrow^{D}$$

$$\left(\widetilde{F}_{j33} + \widetilde{F}_{k33}\right)^{-1}\left(vec(\widetilde{G}_{j3}\Omega^{-1}\alpha_{jI} - \widetilde{G}_{j4}\Omega^{-1}\alpha_{jR}) + vec(\widetilde{G}_{k3}\Omega^{-1}\alpha_{kI} - \widetilde{G}_{k4}\Omega^{-1}\alpha_{kR})\right),$$

In addition,

$$\left(\sum[\hat{X}_{t-1}^{(j)} + \hat{X}_{t-1}^{(k)}]\hat{\Omega}^{-1}[\hat{X}_{t-1}'^{(j)} + \hat{X}_{t-1}'^{(k)}]\right)^{1/2}\left(\hat{\boldsymbol{\beta}}_{jR} - \mathbf{b}\right) \rightarrow^{D} N(\mathbf{0}, I_{r_j d_j})$$

where $\hat{X}_{t-1}'^{(j)} + \hat{X}_{t-1}'^{(k)} = \hat{\alpha}_{jI} \otimes \mathbf{w}_{2t-1}'^{(j)} - \hat{\alpha}_{jR} \otimes \mathbf{v}_{2t-1}'^{(j)} + \hat{\alpha}_{kI} \otimes \mathbf{w}_{2t-1}'^{(k)} - \hat{\alpha}_{kR} \otimes \mathbf{v}_{2t-1}'^{(k)}$ *is a* $m \times r_j d_j$ *matrix,* $\hat{\Omega}$ *is a consistent estimator of* Ω.

Finally, we consider the case: $H_0 : (\boldsymbol{\beta}'_{jR}, \boldsymbol{\beta}'_{jI})' = (\mathbf{b}'_R, \mathbf{b}'_I)'$ in the presence of common PCIV's at different frequencies. The existence of common PCIV's means $(\boldsymbol{\beta}'_{jR}, \boldsymbol{\beta}'_{jI})' = (\boldsymbol{\beta}'_{kR}, \boldsymbol{\beta}'_{kI})'$ for $j \neq k$.

Theorem 3. Under the assumptions of Theorem 1 and in the presence of common polynomial cointegration corresponding to nonstationary roots $\theta_j, \theta_k \in (0, \pi)$ *for* $j \neq k$, *the asymptotic distribution of* $(\hat{\boldsymbol{\beta}}_{jR}, \hat{\boldsymbol{\beta}}_{jI})$ *is, when the correct model with* $(\boldsymbol{\beta}'_{jR}, \boldsymbol{\beta}'_{jI})' = (\boldsymbol{\beta}'_{kR}, \boldsymbol{\beta}'_{kI})'$ *is fitted,*

$$T\begin{pmatrix} \hat{\boldsymbol{\beta}}_{jR} - \mathbf{b}_R \\ \hat{\boldsymbol{\beta}}_{jI} - \mathbf{b}_I \end{pmatrix} \rightarrow^{D} \begin{pmatrix} \widetilde{F}_{j33} + \widetilde{F}_{k33} & \widetilde{F}_{j34} + \widetilde{F}_{k34} \\ \widetilde{F}_{j43} + \widetilde{F}_{k43} & \widetilde{F}_{j44} + \widetilde{F}_{k44} \end{pmatrix}^{-1} \widetilde{G}(j,k)$$

with $\widetilde{G}(j,k) = \begin{pmatrix} vec(\widetilde{G}_{j3}\Omega^{-1}\alpha_{jI} - \widetilde{G}_{j4}\Omega^{-1}\alpha_{jR}) + vec(\widetilde{G}_{k3}\Omega^{-1}\alpha_{kI} - \widetilde{G}_{k4}\Omega^{-1}\alpha_{kR}) \\ vec(\widetilde{G}_{j3}\Omega^{-1}\alpha_{jR} + \widetilde{G}_{j4}\Omega^{-1}\alpha_{jI}) + vec(\widetilde{G}_{k3}\Omega^{-1}\alpha_{kR} + \widetilde{G}_{k4}\Omega^{-1}\alpha_{kI}) \end{pmatrix}$.

In addition,

$$\left(\sum[\hat{X}_{t-1}^{(j)} + \hat{X}_{t-1}^{(k)}]\hat{\Omega}^{-1}[\hat{X}_{t-1}'^{(j)} + \hat{X}_{t-1}'^{(k)}]\right)^{1/2}\begin{pmatrix} \hat{\boldsymbol{\beta}}_{jR} - \mathbf{b}_R \\ \hat{\boldsymbol{\beta}}_{jI} - \mathbf{b}_I \end{pmatrix} \rightarrow^{D} N(\mathbf{0}, I_{2r_j d_j})$$

where $X_{t-1}'^{(j)} + X_{t-1}'^{(k)} = [\alpha_{jI} \otimes \mathbf{w}_{2t-1}'^{(j)} - \alpha_{jR} \otimes \mathbf{v}_{2t-1}'^{(j)} + \alpha_{kI} \otimes \mathbf{w}_{2t-1}'^{(k)} - \alpha_{kR} \otimes \mathbf{v}_{2t-1}'^{(k)}$, $\alpha_{jR} \otimes \mathbf{w}_{2t-1}'^{(j)} + \alpha_{jI} \otimes \mathbf{v}_{2t-1}'^{(j)} + \alpha_{kR} \otimes \mathbf{w}_{2t-1}'^{(k)} + \alpha_{kI} \otimes \mathbf{v}_{2t-1}'^{(k)}]$ *is a* $m \times 2r_j d_j$ *matrix,* $\hat{\Omega}$ *is a consistent estimator of* Ω.

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