STRONG CONVERGENCE FOR WEIGHTED SUMS OF FUZZY RANDOM VARIABLES

YUN KYONG KIM[1]

ABSTRACT. In this paper, we establish some results on strong convergence for weighted sums of uniformly integrable fuzzy random variables taking values in the space of upper-semicontinuous fuzzy sets in R^p.

1. Introduction

Pruitt (1966) obtained almost sure convergence for weighted sums of independent identically distributed real-valued random variables $\{X_n\}$ by assuming

$$\max_{1\le k\le n} |a_{nk}| = O(n^{-\gamma}) \quad \text{for some } \gamma > 0$$

and

$$E|X_1|^{1+\frac{1}{\gamma}} < \infty.$$

Rohatgi (1971) extended Pruitt's result to independent, but not necessarily identically distributed random variables $\{X_n\}$ by requiring stochastically bounded condition, i.e., there exists a random variable X with $E|X|^{1+\frac{1}{\gamma}} < \infty$ such that for each n,

$$P(|X_n| \ge \lambda) \le P(|X| \ge \lambda) \quad \text{for all } \lambda > 0.$$

Taylor and Inoue (1985) showed that the similar results can be obtained for random sets.

Key words and phrases. Fuzzy random variables, Strong law of large numbers, uniform integrability, tightness, weighted sums..
[1]Department of Information & Communication Engineering, Dongshin University

The purpose of this paper is to generalize the results of Taylor and Inoue (1985) to the fuzzy case.

2. Preliminaries

Let $\mathcal{K}(R^p)$ denote the family of non-empty compact subsets of the Euclidean space R^p. Then the space $\mathcal{K}(R^p)$ is metrizable by the Hausdorff metric h defined by

$$h(A, B) \;=\; \max\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\}.$$

A norm of $A \in \mathcal{K}(R^p)$ is defined by

$$\|A\| = h(A, \{0\}) = \sup_{a\in A}|a|.$$

It is well-known that $\mathcal{K}(R^p)$ is complete and separable with respect to the Hausdorff metric h. The addition and scalar multiplication on $\mathcal{K}(R^p)$ are defined as usual:

$$\begin{aligned} A \oplus B &= \{a+b : a \in A, b \in B\} \\ \lambda A &= \{\lambda a : a \in A\} \end{aligned}$$

for $A, B \in \mathcal{K}(R^p)$ and $\lambda \in R$.

Let $\mathcal{F}(R^p)$ denote the family of all fuzzy sets $\tilde{u} : R^p \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;
(2) $\tilde{u}$ is upper semicontinuous;
(3) $supp\,\tilde{u} = cl\{x \in R^p : \tilde{u}(x) > 0\}$ is compact, where cl denotes the closure.

For a fuzzy subset $\tilde{u}$ of R^p, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \text{if } \alpha = 0. \end{cases}$$

Then, it follows immediately that

$$\tilde{u} \in \mathcal{F}(R^p) \text{ if and only if } L_\alpha \tilde{u} \in \mathcal{K}(R^p) \text{ for each } \alpha \in [0,1].$$

Also, if we denote $cl\{x \in R^p : \tilde{u}(x) > \alpha\}$ by $L_{\alpha^+}\tilde{u}$, then

$$\lim_{\beta \downarrow \alpha} h(L_\beta \tilde{u}, L_{\alpha^+}\tilde{u}) = 0.$$

The linear structure on $\mathcal{F}(R^p)$ is also defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0, \\ \tilde{0}(z), & \lambda = 0, \end{cases}$$

for $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$. Recall that a fuzzy subset $\tilde{u}$ of R^p is said to be convex if $\tilde{u}(\lambda x+(1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R^p$ and $\lambda \in [0,1]$. The convex hull of of $\tilde{u}$ is defined by

$$co(\tilde{u}) = \inf\{\tilde{v} : \tilde{v} \text{ is convex and } \tilde{v} \geq \tilde{u}\}.$$

Now we define the metric d_∞ on $\mathcal{F}(R^p)$ as usual;

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}).$$

Also, the norm of $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \|L_0 \tilde{u}\| = \sup_{x \in L_0 \tilde{u}} |x|.$$

3. Main Results

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space. A set-valued function $X : \Omega \rightarrow \mathcal{K}(R^p)$ is called a random set if it is measurable. A random set X is said to be integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set X is defined by

$$E(X) = \{E(\xi) : \xi \in L(\Omega, R^p) \, \text{and} \, \xi(\omega) \in X(\omega) \, a.s.\},$$

where $L(\Omega, R^p)$ denotes the class of all R^p-valued random variables ξ such that $E|\xi| < \infty$.

A fuzzy set valued function $\tilde{X} : \Omega \to \mathcal{F}(R^p)$ is called a fuzzy random variable if for each $\alpha \in [0,1], L_\alpha \tilde{X}$ is a random set. This definition was introduced by Puri and Ralescu (1986) as a natural generalization of a random set.

A fuzzy random variable $\tilde{X}$ is said to be integrably bounded if $E\|\tilde{X}\| < \infty$. The expectation of integrably bounded fuzzy random variable $\tilde{X}$ is a fuzzy subset $E(\tilde{X})$ of R^p defined by

$$E(\tilde{X})(x) = \sup\{\alpha \in [0,1] : x \in E(L_\alpha \tilde{X})\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of integrably bounded fuzzy random variables and $\{a_{ni}\}$ be a Toeplitz sequence, i.e., $\{a_{ni}\}$ is a double array of real numbers satisfying

(1) For each i, $\lim_{n\to\infty} a_{ni} = 0$;

(2) For each $n, \sum_{i=1}^{\infty} |a_{ni}| \leq C$ for some $C > 0$.

The problem which we will consider is to find sufficient conditions for strong convergence of weighted sums of $\{\tilde{X}_n\}$ in the following;

$$d_\infty(\oplus_{i=1}^n a_{ni}\tilde{X}_i, \oplus_{i=1}^n a_{ni} co(E\tilde{X}_i)) = 0 \quad a.s.,$$

where $co(E\tilde{X}_i)$ denotes the convex hull of $E(\tilde{X}_i)$.

To this end, we require the following assumption:

(A): For each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,

$$\max_{1\leq k\leq m} Eh(L_{\alpha_{k-1}^+}\tilde{X}_n, L_{\alpha_k}\tilde{X}_n) < \epsilon.$$

The next theorem implies that if $\{\tilde{X}_n\}$ is identically distributed, then it satisfies the condition (A).

Theorem 3.1. *Let $E\|\tilde{X}\| < \infty$. Then for each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,*

$$\max_{1\leq k\leq m} Eh(L_{\alpha_{k-1}^+}\tilde{X}, L_{\alpha_k}\tilde{X}) < \epsilon.$$

Theorem 3.2. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying (A). Suppose that the following two conditions are satisfied;*

(1) *$\{\|\tilde{X}_n\|\}$ is uniformly integrable.*
(2) *There exists a nonnegative random variable X such that for each n,*

$$P(\|\tilde{X}_n\| \geq \lambda) \leq P(X \geq \lambda) \quad \text{for all } \lambda > 0,$$

and $EX^{1+\frac{1}{\gamma}} < \infty$ for some $\gamma > 0$.

If $\{a_{ni}\}$ is a Toeplitz sequence satisfying $\max_{1 \leq i \leq n} |a_{ni}| = O(n^{-\gamma})$, then

$$d_\infty(\oplus_{i=1}^n a_{ni}\tilde{X}_i, \oplus_{i=1}^n a_{ni} co(E\tilde{X}_i)) = 0 \quad a.s.$$

Corollary 3.3. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying (A). Suppose that there exists a compact subset K of $\mathcal{K}(R^p)$ such that $P\{L_0\tilde{X}_n \in K\} = 1$ for all n.*

If $\{a_{ni}\}$ is a Toeplitz sequence satisfying $\max_{1 \leq i \leq n} |a_{ni}| = O(n^{-\gamma})$ for some $\gamma > 0$, then

$$d_\infty(\oplus_{i=1}^n a_{ni}\tilde{X}_i, \oplus_{i=1}^n a_{ni} co(E\tilde{X}_i)) = 0 \quad a.s.$$

Note that Corollary 3.3 provide a SLLN by choosing $a_{ni} = 1/n, 1 \leq i \leq n$; $a_{ni} = 0, i > n$. While on the other hand, we need the restrictive condition $\sup_n E\|\tilde{X}_n\|^2 < \infty$ in order to provide a SLLN by applying Theorem 3.2. But we can obtain much better results by similar arguments in the proof of Theorem 3.2.

Theorem 3.4. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying (A). Suppose that the following two conditions are satisfied;*

(1) *$\{\|\tilde{X}_n\|\}$ is uniformly integrable.*
(2) *$\sum_{n=1}^\infty \frac{1}{n^r} E\|\tilde{X}_n\|^r < \infty$ for some $1 \leq r \leq 2$.*

Then

$$\frac{1}{n} d_\infty(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n co(E\tilde{X}_i)) = 0 \quad a.s.$$

We can obtain the following theorem from the fact that tightness and r-th ($r > 1$) moments condition of $\{L_0\tilde{X}_n\}$ imply uniform integrability of $\{\|\tilde{X}_n\|\}$.

Corollary 3.5. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying (A). Suppose that $\{L_0\tilde{X}_n\}$ is tight random sets and*

$$\sup_n E\|\tilde{X}_n\|^r < \infty \quad \text{for some } r > 1.$$

Then

$$\frac{1}{n}d_\infty(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n co(E\tilde{X}_i)) = 0 \quad a.s.$$

References

[1]. Z. Artstein and J. C. Hansen, *Convexification in limit laws of random sets in Banach spaces*, Ann. Probab. **13** (1985), 307-309.

[2]. A. Colubi, J. S. Dominguez-Menchero, M. Lopez-Diaz, R. Körner, *A method to derive strong laws of large numbers for random upper semicontinuous functions*, Statistics & Probab. Lett. **53** (2001), 269-275.

[3]. Y. Feng, *An approach to generalize law of large numbers for fuzzy random variables*, Fuzzy Sets and Systems **128** (2002), 237-245.

[4]. H. Inoue, *A strong law of large numbers for fuzzy random sets*, Fuzzy Sets and Systems **41** (1991), 285-291.

[5]. S. Y. Joo and Y. K. Kim, *Kolmogorov's strong law of large numbers for fuzzy random variables*, Fuzzy Sets and Systems **120** (2001), 499-503.

[6]. Y. K. Kim, *Strong law of large numbers for random upper semicontinuous functions*, Bull. Kor. Math. Soc. **39** (2002), 511-526.

[7]. E. P. Klement, M. L. Puri and D. A. Ralescu, *Limit theorems for fuzzy random variables*, Proc. Roy. Soc. Lond. Ser. A **407** (1986), 171-182.

[8]. W. Pruitt, *Summability of independent random variables*, J. Math. & Mech. Appl. **15** (1966), 769-776.

[9]. M. L. Puri and D. A. Ralescu, *Fuzzy random variables*, Jour. Math. Anal. Appl. **114** (1986), 402-422.

[10].V. K. Rohatgi, *Convergence of weighted sums of independent random variables*, Proc. Cambridge Philos. Soc. **69** (1971), 305-307.

[11].R. L. Taylor and H. Inoue,, *A strong law of large numbers for random sets in Banach space*, Bull. Inst. Math. Academia Sinica **13** (1985), 403-409.

[12].R. L. Taylor and H. Inoue,, *Convergence of weighted sums of random sets*, Stochastic Anal. Appl. **3** (1985), 379-396.

[13].T. Uemura, *A law of large numbers for random sets,*, Fuzzy Sets and Systems **59** (1993), 181-188.