A Distribution of Terminal Time Value and Running Maximum of Two-Dimensional Brownian Motion with an Application to Barrier Option

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Abstract

This presentation derives a distribution function of the terminal value and running maximum of two-dimensional Brownian motion $\{X(t) = (X_1(t), X_2(t))', t > 0\}$. One random variable of the joint distribution is the terminal time value of the Brownian motion $\{X_1(t), t > 0\}$. The other random variable is the partial-time running maximum of the Brownian motion $\{X_2(t), t > 0\}$. With this distribution function, this presentation also derives an explicit pricing formula for a barrier option whose monitoring period of the option starts at an arbitrary date and ends at another arbitrary date before maturity.

Key words: Brownian motion, running maximum, terminal time value, barrier option

1. Introduction

Merton (1973) and Reiner and Rubinstein (1991) have developed pricing formulas for standard barrier options. The word "standard" means that the monitoring period is the entire option life. Heynen and Kat (1994b) derived pricing formulas for barrier options whose monitoring periods are $[0, t]$ or $[t, T]$ instead of the entire option life, $[0, T]$. Heynen and Kat (1994a) derived pricing formulas for outside barrier options whose monitoring period is $[0, T]$. Bermin (1996) developed explicit pricing formulas for outside barrier options with the monitoring period from time 0 to time t $(t < T)$.

This paper derives a distribution function of the terminal value and running maximum of two-dimensional Brownian motion $\{(X_1(t), X_2(t))', t > 0\}$. One random variable of the joint distribution is the terminal time value of the Brownian motion $\{X_1(t), t > 0\}$. The other random variable is the partial-time running maximum of the Brownian motion $\{X_2(t), t > 0\}$. With this distribution function, this paper also derives an explicit pricing formula for a barrier options whose monitoring period of the option starts at an arbitrary date and ends at another arbitrary date before maturity.

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2. Two-Dimensional Brownian Motion and its Distributions

Consider one-dimensional Brownian motion $\{X(\tau), \tau \geq 0\}$ with drift μ and volatility parameter σ. Thus, $X(\tau)$ has a normal distribution with mean μt and variance $\sigma^2 t$. Let

$$M(s, t) = \max\{X(\tau), s \leq \tau \leq t\} \tag{2.1}$$

be the maximum of the Brownian motion between time s and time t. For $0 < s < t < T$, the joint probability distribution function of $X(T)$ and $M(s, t)$ is

$$\begin{aligned}
&\Pr(X(T) \leq x, M(s, t) \leq m) \\
&= \Phi_3\left(\frac{x-\mu T}{\sigma\sqrt{T}}, \frac{m-\mu t}{\sigma\sqrt{t}}, \frac{m-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\
&\quad - e^{\frac{2\mu}{\sigma^2}m}\,\Phi_3\left(\frac{x-2m-\mu T}{\sigma\sqrt{T}}, \frac{-m-\mu t}{\sigma\sqrt{t}}, \frac{m+\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right),
\end{aligned} \tag{2.2}$$

which can be shown in Lee (2002). Here, $\Phi_3(\ )$ denotes a trivariate standard normal distribution function.

Next, let us consider a two-dimensional Brownian motion $\{X(t) = (X_1(t), X_2(t))'\}$ with drift vector $(\mu_1, \mu_2)'$, $X_i(0) = 0$ and diffusion matrix equal to

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

For $0 < s < t$, let

$$M_2(s, t) = \max\{X_2(\tau), s \leq \tau \leq t\} \tag{2.3}$$

be the maximum of the Brownian motion $\{X_2(\tau), 0 \leq \tau\}$ between time s and time t. In Section 3, we shall prove that for $0 < s < t \leq T$, the joint distribution function of $M_2(s, t)$ and $X_1(T)$ is

$$\begin{aligned}
&\Pr(X_1(T) \leq x, M_2(s, t) \leq m) \\
&= \Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{m-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{m-\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\
&\quad - e^{\frac{2\mu_2}{\sigma_2^2}m}\,\Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho m}{\sigma_2\sqrt{T}}, \frac{-m-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{m+\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right).
\end{aligned} \tag{2.4}$$

If $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\rho = 1$, then the random vector $(X_1(T), M_2(s, t))$ has the same distribution as the random vector $(X_1(T), M_1(s, t))$.

3. Proof of (2.33)

Let us derive the joint distribution function of random variables $X_1(T)$ and $M_2(s, t)$. First, consider the case that the correlation coefficient ρ is nonzero. Let

$$Z(\tau) := \frac{\sigma_2}{\sigma_1} X_1(\tau) - \rho X_2(\tau). \tag{3.1}$$

The random variable $Z(\tau)$ is independent of $X_2(\tau)$ because their covariance is zero. Thus the stochastic processes $\{Z(\tau)\}$ and $\{X_2(\tau)\}$ are independent. The joint distribution function of $X_1(T)$ and $M_2(s, t)$ can be calculated as follows:

$$\Pr(X_1(T) \le x, M_2(s, t) \le m) = \Pr(\frac{\sigma_1}{\sigma_2}(Z(T) + \rho X_2(T)) < x, M_2(s, t) \le m)$$

$$= \mathrm{E}[\Pr(\rho X_2(T) < \frac{\sigma_2}{\sigma_1} x - Z(T), M_2(s, t) \le m \mid Z(T))]. \tag{3.2}$$

Now, let us calculate the inside conditional probability term in (3.2). It follows from the joint distribution of $X_2(T)$ and $M_2(s, t)$ that the inside conditional probability term in (3.2) can be easily obtained. The following probability can be calculated,

$$\Pr(X_2(T) > x, M_2(s, t) < m)$$

$$= \Pr(M_2(s, t) < m) - \Pr(X_2(T) < x, M_2(s, t) < m)$$

$$= \Phi_2(d, e; \sqrt{\frac{s}{t}}) - e^{\frac{2\mu_2}{\sigma_2^2}m} \Phi_2(f, g; -\sqrt{\frac{s}{t}}) - \{\Phi_3(\frac{x - \mu_2 T}{\sigma_2\sqrt{T}}, d, e; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}})$$

$$- e^{\frac{2\mu_2}{\sigma_2^2}m} \Phi_3(\frac{x - 2m - \mu_2 T}{\sigma_2\sqrt{T}}, f, g; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}})\}$$

$$= \Phi_3(-\frac{x - \mu_2 T}{\sigma_2\sqrt{T}}, d, e; -\sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}})$$

$$- e^{\frac{2\mu_2}{\sigma_2^2}m} \Phi_3(-\frac{x - 2m - \mu_2 T}{\sigma_2\sqrt{T}}, f, g; -\sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}), \tag{3.3}$$

where $d = \frac{m - \mu_2 t}{\sigma_2\sqrt{t}}$, $e = \frac{m - \mu_2 s}{\sigma_2\sqrt{s}}$, $f = \frac{-m - \mu_2 t}{\sigma_2\sqrt{t}}$, and $g = \frac{m + \mu_2 s}{\sigma_2\sqrt{s}}$. Here, $\Phi_2(\ )$ denotes a bivariate standard normal distribution function. Applying the probability formulas (2.2) and (3.3), we can obtain the probability formula,

$$\Pr(\rho X_2(T) < x, M_2(s,t) < m)$$
$$= \Phi_3(s(\rho)\frac{x/\rho - \mu_2 T}{\sigma_2\sqrt{T}}, d, e; s(\rho)\sqrt{\frac{t}{T}}, s(\rho)\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}})$$
$$-e^{\frac{2\mu_2}{\sigma_2^2}m}\,\Phi_3(s(\rho)\frac{x/\rho - 2m - \mu_2 T}{\sigma_2\sqrt{T}}, f, g; s(\rho)\sqrt{\frac{t}{T}}, -s(\rho)\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}), \qquad (3.4)$$

where $s(\rho)$ is 1 if ρ is greater than zero and $s(\rho)$ is -1 otherwise. Thus it follows from applying (3.4) that the last line of (3.2) can be rewritten as

$$E[\Phi_3(s(\rho)\frac{(\frac{\sigma_2}{\sigma_1}x - Z(T))/\rho - \mu_2 T}{\sigma_2\sqrt{T}}, d, e; s(\rho)\sqrt{\frac{t}{T}}, s(\rho)\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}})]$$
$$-e^{\frac{2\mu_2}{\sigma_2^2}m}\,E[\Phi_3(s(\rho)\frac{(\frac{\sigma_2}{\sigma_1}x - Z(T))/\rho - 2m - \mu_2 T}{\sigma_2\sqrt{T}}, f, g; s(\rho)\sqrt{\frac{t}{T}}, -s(\rho)\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}})]. \qquad (3.5)$$

Consider the first expectation in (3.5). Let (U, V, W) be a random vector with trivariate standard normal distribution and correlation coefficients $\mathrm{Corr}(U, V) = s(\rho)\sqrt{\frac{t}{T}}$, $\mathrm{Corr}(U, W) = s(\rho)\sqrt{\frac{s}{T}}$ and $\mathrm{Corr}(V, W) = \sqrt{\frac{s}{t}}$. Assume that the random variable $Z(T)$ is independent of the random vector (U, V, W). Then, the first expectation in (3.5) can be calculated as follows:

$$E[E[I(U < s(\rho)\frac{(\frac{\sigma_2}{\sigma_1}x - Z(T))/\rho - \mu_2 T}{\sigma_2\sqrt{T}}; V < d, W < e) \mid Z(T)]]$$
$$= E[I(|\rho|\sigma_2\sqrt{T}\,U + Z(T) < \frac{\sigma_2}{\sigma_1}x - \rho\mu_2 T, V < d, W < e)]$$
$$= \Phi_3(\frac{x - \mu_1 T}{\sigma_1\sqrt{T}}, d, e; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}). \qquad (3.6)$$

Let us calculate the second expectation in (3.5). Assume that the random vector (U, V, W) has a trivariate standard normal distribution with correlation coefficients $\mathrm{Corr}(U, V) = s(\rho)\sqrt{\frac{t}{T}}$, $\mathrm{Corr}(U, W) = -s(\rho)\sqrt{\frac{s}{T}}$ and $\mathrm{Corr}(V, W) = -\sqrt{\frac{s}{t}}$. Also assume that the random variable $Z(T)$ is independent of the random vector (U, V, W). Then, the second expectation will be calculated as follows:

$$\mathrm{E}[\mathrm{E}[\mathrm{I}(U < s(\rho)\frac{(\frac{\sigma_2}{\sigma_1}x - Z(T))/\rho - 2m - \mu_2 T}{\sigma_2\sqrt{T}},\ V<f,\ W<g) \mid Z(T)]]$$

$$= \mathrm{E}[\mathrm{I}(|\rho|\sigma_2\sqrt{T}\,U + Z(T) < \frac{\sigma_2}{\sigma_1}x - 2\rho m - \rho\mu_2 T,\ V<f,\ W<g)]$$

$$= \Phi_3(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho m}{\sigma_2\sqrt{T}}, f, g; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}). \tag{3.7}$$

Finally, it is straightforward to consider the case that the correlation coefficient ρ is zero, because the stochastic processes $\{X_1(\tau)\}$ and $\{X_2(\tau)\}$ are independent.

4. Application to an Outside Barrier Option

This section applies the joint distribution function of $X_1(T)$ and $M_2(s, t)$ to derive an explicit pricing formula for outside barrier options whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity. The payoffs of the outside barrier options depend on prices of two underlying assets. Let $S_1(t)$ and $S_2(t)$ denote the time-t prices of two underlying assets. Assume that these assets pay no dividends. Assume that for $t \geq 0$, $i = 1$ and 2,

$$S_i(t) = S_i(0)\exp(X_i(t)),$$

where $\{(X_1(t), X_2(t))'\}$ is a 2-dimensional Brownian motion as mentioned in Section 2.

Let us take a look at an up-and-out outside barrier put option. Assume that the strike price is K, and the barrier level is B. Let $b = \log[B/S_2(0)]$ and $k = \log[K/S_1(0)]$. The activating condition of the barrier option is $\{M_2(s, t) < b\}$. The put condition is $\{X_1(T) < k\}$. The payoff of the put option will be $(K - S_1(T))$ if the option satisfies its activating condition and $S_1(T)$ is less than K. The payoff can be expressed as follows:

$$\begin{aligned} &K - S_1(T),\ \text{if } M_2(s, t) < b \text{ and } X_1(T) < k \\ &0, \text{ otherwise}, \end{aligned} \tag{4.1}$$

By the fundamental theorem of asset pricing and by Esscher transforms of Geber and Shiu (1996), the time-0 value of the payoff (4.1) is

$$e^{-rT}\mathrm{E}^*[-(S_1(T) - K)\mathrm{I}(M_2(s, t) < b, X_1(T) < k)]$$

$$= -S_1(0)\Pr{}^{**}(M_2(s, t) < b, X_1(T) < k) + e^{-rT}K\Pr{}^{*}(M_2(s, t) < b, X_1(T) < k). \tag{4.2}$$

Now, the final step for pricing the outside barrier option is to calculate the probabilities of (4.2). These probabilities are the same as (2.4) except that the drift parameter vectors of the first and second probabilities are

$$(\mu_1^{**}, \mu_2^{**}) = (r + \sigma_1^2/2, r - \sigma_2^2/2 + \rho\sigma_1\sigma_2) \tag{4.3a}$$

and

$$(\mu_1^{*}, \mu_2^{*}) = (r - \sigma_1^2/2, r - \sigma_2^2/2), \tag{4.3b}$$

respectively. Similarly, we may derive pricing formulas for several types of outside barrier options.

References

Bermin, H. P. 1996. "Time and Path Dependent Options: The Case of Time Dependent Inside and Outside Barrier Options," *Paper presented at the Third Nordic Symposium on Contingent Claims Analysis in Finance, Iceland, May.*

Gerber, H.U. and Shiu, E.S.W. 1996. "Actuarial Bridges to Dynamic Hedging and Option Pricing," *Insurance: Mathematics and Economics* 18:183-218.

Heynen, R.C. and Kat, H.M. 1994a. "Crossing Barriers," *Risk* 7 (6):46-50.

Heynen, R.C. and Kat, H.M. 1994b. "Partial Barrier Options," *The Journal of Financial Engineering* 3(4):253-274.

Lee, H. 2002. *Pricing Exotic Options with Applications to Equity-Indexed Annuities*, Doctoral Thesis, The University of Iowa.

Merton, R. (1973). "The Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science* 4: 141-183.

Reiner, E. and Rubinstein, M. 1991. "Breaking Down the Barrier," *RISK* 4(8): 28-35