한국전산구조공학회
2003년도 가을 학술발표회 논문집

비압축성 2-D 유동에 대한 와도-흐름함수 방정식의 유한요소 근사

FE Approximation of the Vorticity–Stream function Equations for Incompressible 2-D flows

박 성 관[*] 김 도 완[**] 권 영 철[***]
Pak, Seong-Kwan Kim, Do-Wan Kweon, Young-Cheol

ABSTRACT

The object of this paper is the treatment of how to make the vorticity boundary condition instead of pressure in the primitive variable case. An improved algorithm for solving the vorticity-stream function equation is presented. The linear finite element approximation for the solution of Navier-Stokes and Stokes flows is constructed. Not only regular domain but also complicate domain can be analyze d, using this formulation.

1. Introduction

In this paper, we present a very simple and efficient finite element method for 2-D viscous incompressible flows using the stream-vorticity formulation. In 2-D incompressible flow simulation, the stream-vorticity formulation[(1)] is often used. The main advantage is a reduction from three partial differential equations for the primitive variables(pressure and two velocity components) to either one(4^{th} order) partial differential equation for the stream function or to a set of two(2^{nd} order) equations for the scalar quantities vorticity and stream function. Another advantage is that the continuity equation will be satisfied automatically. On the other hands, the value of the vorticity at no-slip boundary conditions is difficult to specify and a poor evaluation of this boundary condition leads, almost invariably, to serious difficulties in obtaining a converged solutions. Traditional finite element approaches based on finite difference formulation for the wall vorticity are generally limited to regular domains to low-Reynolds-number flows.

Therefore, in recent years, a lot of effort has been devoted to the consistent specification of the vortictiy boundaries, in the context of both Galerkin and control-volume-based finite element methods.[(2)] A guideline for a correct specification of boundary conditions at no-slip walls has been given by Glowinski and Pironneau.[(3)]

For the implementation of the vorticity boundary condition in the context of the finite element method, various approaches have been followed, ranging from fully integrated procedures. In Reference (2), the stream function, the wall voriticity and the field vorticity are computed simultaneously. In Reference (4), the stream function and the wall vorticity are first solved simultaneously and then the field vorticity is solved separately. In this paper, we solve first the stream function, and then compute the wall voriticity and finally compute the field

* 연세대학교 사회환경시스템 공학부 박사과정
** 선문대학교 수학과 부교수
*** 선문대학교 기계공학과 조교수

vorticity. In this way we generalize the results of Reference (2), Following a fully segregated approach in which the three differential equations are completely uncoupled and dealt with in sequence. In the discretization of the stream function equation, moreover, our approach leads to smaller matrices and preserves the symmetry, which is lost in Reference (4) due to the simultaneous solution of stream function and wall vorticity.

In the finite element formulation we rely on the Bubnov-Galerkin method for space discretization, without using any upwinding technique. Thus we reach convergence, even for very high values of the Reynolds number, without exploiting the effects of the numerical viscosity introduced by most upwinding procedures. As a numerical example, we calculate the numerical solutions for the Stokes and the Navier-Stokes equations in 2-D. The stability and accuracy[5] of our approach are demonstrated by the solution of well-known benchmark problem and the several plots for the numerical solution are shown in this paper.

2. Problem Statement

The simplest formulation of the Naiver-Stokes equations for 2-D incompressible laminar flows is in terms of the stream function, ψ, and the vorticity, ω. The vorticity-stream function formulation gives two equations, on an advection-diffusion equation for the vorticity and the other a relation between the vorticity and the stream function.[6]

The vorticity equation is

$$\frac{\partial \omega}{\partial t} + Re\left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} \right) = \Delta\omega + f \tag{1}$$

where the domain is some part of (x, y) space. f is body force and Re is the Reynolds number. The fluid velocities are defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{2}$$

where the stream function, ψ, is connected to the vorticity, ω, by

$$\begin{cases} \omega = -\Delta\psi \\ \psi|_{\partial\Omega} = g \end{cases} \tag{3}$$

where $\omega = \nabla \times \mathbf{u} = -u_y + v_x$ is the vorticity, ψ is the stream function formulation which satisfies the no-slip boundary condition

$$\psi_{\mathbf{n}}|_{\partial\Omega} = \frac{\partial \psi}{\partial \mathbf{n}} = q \tag{4}$$

in addition to the no-penetration boundary condition $\psi_{\mathbf{n}}|_{\partial\Omega} = q$, where $\mathbf{n}$ is the unit normal vector of the boundary $\partial\Omega$ pointing outward. Adding inhomogeneous terms to the boundary condition only amounts to minor changes in what follows.

We uncouple these equations in Reference (2) into three second order elliptic ones such that

$$\begin{cases} -\Delta\psi_g = 0 \\ \psi_g|_\Gamma = g \end{cases} \tag{5}$$

$$\begin{cases} -\Delta\psi = \omega \\ \psi|_\Gamma = 0 \end{cases} \tag{6}$$

$$\begin{cases} \dfrac{\partial\omega}{\partial t} + Re(\mathbf{u}\cdot\nabla)\omega = \Delta\omega \\ \omega|_\Gamma = \alpha \end{cases} \tag{7}$$

where α is not known apriori but is implicitly subjected to

$$\frac{\partial\psi}{\partial\mathbf{n}}|_\Gamma = q - \frac{\partial\psi_g}{\partial\mathbf{n}}. \tag{8}$$

We are prepared to discretize the above decoupling system using finite element approximation. Let V^h be the usual linear finite element space for both ψ and ω which is defined by

$$V^h = \{N_h \in C(\Omega) | N_h|_\tau \in P_1 \text{ for all } \tau \in T_h\}$$

where T_h is the triangulation for Ω, h represents mesh size and P_1 stands for the set of polynomials of linear order. Also $\{\phi_i | i = 1,2,\cdots,n\}$ designates the basis for the finite dimensional space V^h. Here we decompose the space V^h into interior part and boundary one such that

$$V^h = V_I^h + V_B^h. \tag{9}$$

In fact, we can deduce

$$V_I^h = \left\{\sum_{i\in\Lambda_I} c_i\phi_i \middle| \Lambda_I \text{ is the index set of basis functions for interior nodes}\right\} \tag{10}$$

$$V_B^h = \left\{\sum_{i\in\Lambda_B} c_i\phi_i \middle| \Lambda_B \text{ is the index set of basis functions for boundary nodes}\right\}. \tag{11}$$

We propose the iterative method to solve the Navier-Stokes equation using the decoupling system (6), (7). The second equation (7) becomes well-posed if its boundary condition α is available.

This is the main step of our algorithm.

STEP 0: Calculate the solution $\psi_g^h \in \{N_h \in V^h | N_h|_\Gamma = g\}$ of the equation.

$$\int_\Omega \nabla\psi_g^h \nabla N_h^I \, d\Omega = 0 \qquad \text{for all } N_h^I \in V_I^h \tag{12}$$

STEP 1: Assume that $\omega_h^{(n)}$ is given in Ω.

Calculate the solution $\psi_h^{(n)} \in \{N_h \in V^h | N_h|_\Gamma = 0\}$ of the equation.

$$\int_\Omega \nabla \psi_h^{(n)} \nabla N_h^I \, d\Omega = \int_\Omega \omega_h N_h^I \, d\Omega \qquad \text{for all } N_h^I \in V_I^h \tag{13}$$

Then we separate $\omega_h^{(n)}$ into two parts such that

$$\omega_h^{(n)} = \omega_h^{I(n)} + \omega_h^{B(n)} \tag{14}$$

STEP 2: Let $\omega_h^{B(n+1)} = \sum_{i \in \Lambda_B} \omega_i^B \phi_i$.

Calculate the coefficient ω_i^B satisfying the equation

$$\int_\Omega \omega_h^{B(n+1)} N_h^B \, d\Omega = -\int_{\partial\Omega} \left(q - \frac{\partial \psi_g}{\partial \mathbf{n}} \right) N_h^B \, d\Gamma + \int_\Omega \nabla \psi_h^{(n)} \nabla N_h^B \, d\Omega$$
$$- \int_\Omega \omega_h^{I(n)} N_h^B \, d\Omega \qquad \text{for all } N_h^B \in V_B^h \tag{15}$$

STEP 3: Set the vorticity boundary condition

$$\alpha^{(n)} = \omega_h^{B(n)}|_\Gamma + \gamma \left(\omega_h^{B(n+1)}|_\Gamma - \omega_h^{B(n)}|_\Gamma \right)$$

If $\max \left\| \frac{\left(\omega_h^{B(n+1)}|_\Gamma - \omega_h^{B(n)}|_\Gamma \right)}{\omega_h^{B(n)}|_\Gamma} \right\|$ is small enough, then STOP.

Otherwise, calculate the solution $\omega_h^{(n+1)} \in \{N_h \in V^h | N_h|_\Gamma = \alpha^{(n)}\}$ of the equation.

$$\int_\Omega \left(\frac{\omega_h^{(n+1)} - \omega_h^{(n)}}{\Delta t} \right) N_h^I \, d\Omega + \int_\Omega \nabla \omega_h^{(n+1)} \nabla N_h^I \, d\Omega$$
$$+ Re \int_\Omega \left(\frac{\partial \psi_h^{(n)}}{\partial y} \frac{\partial \omega_h^{(n+1)}}{\partial x} - \frac{\partial \psi_h^{(n)}}{\partial x} \frac{\partial \omega_h^{(n+1)}}{\partial y} \right) N_h^I \, d\Omega = 0 \qquad \text{for all } N_h^I \in V_I^h \tag{16}$$

GOTO STEP 1.

3. Numerical Result

The examples presented here concern steady state solution of the 2-D lid-driven cavity flow. Since we are only interested in the stationary results obtained from pseudo transient simulations, the fully implicit algorithm for time integration is always used. As a convergence criterion, we consider the difference in the field vorticity computed and terminate the calculation when a convergence tolerance of 0.0001 in the normalized residual norm.

The system of algebraic equations obtained from the finite element discretizations are solved by means of iterative procedures derived from the family of preconditioned conjugate gradient(CG) and preconditioned conjugate residual(CR) methods. We use a non-symmetric biconjugate gradient stabilized(Bi-CGSTAB) solver for the vorticity equation and a symmetric CG solver for the stream function equation.[7]

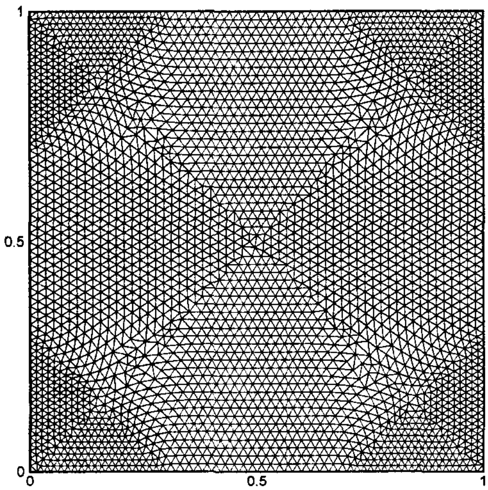


Figure 1. Finite Element Mesh in Cavity

3.1. Flow in a lid-driven cavity

Viscous fluid flow in the driven cavity has long been a popular test case for evaluating numerical techniques. The problem statement is straightforward, the geometry simple, yet the governing equations are non-linear system of partial differential equations.

Our approach is demonstrated by the solution of the 2-D cavity flow. The mesh used in these computations consists of 6180 elements and 3211 nodes, and is shown in Figure 1. Figure 2 (a) and (b) are stream line and vorticity for Stokes (Re=0). Figure 2 (c) and (d) are the streamline and vorticity for Navier-Stokes (Re=100). Figure 2 (e) and (f) are the streamline and vorticity for Navier-Stokes (Re=1000). Figures show that the symmetry of our numerical solution is well presented, which is intrinsic property of the Stokes flow in a symmetric domain. The solution for Navier-Stokes flow is also well illustrated. These numerical results show a good agreement with those of Reference (5).

4. Conclusions

The incompressible 2-D Navier-Stokes equations have been solved by the finite element method using a new vorticity-stream function formulation. The stream function and vorticity equations have been completely uncoupled and solved in sequence, taking advantage of a procedure that fits naturally in the framework of finite element techniques and allows an easy evaluation of vorticity at no-slip boundaries. The present scheme achieves convergence, even for very high values of the Reynolds number, without the traditional need for upwinding.

This method can be applied resolve not only regular domains but also complicate domains. From the present study, we can find that the vorticity-stream function formulation seems recommendable for the incompressible 2-D Navier-Stokes and the Stokes flows.

Reference

1. Peyret, R. and Talyor, T. D., Computational method for fluid flow, Sprin ger, Berlin, 1983.

2. Camprion-Renson, A. and Crochet, M. J., "On the stream function-vorticity finite element solutions of Navier-Stokes equations," *International Journal for Numerical Methods in Engineering*, Vol.12, 1978, pp.1809-1818.

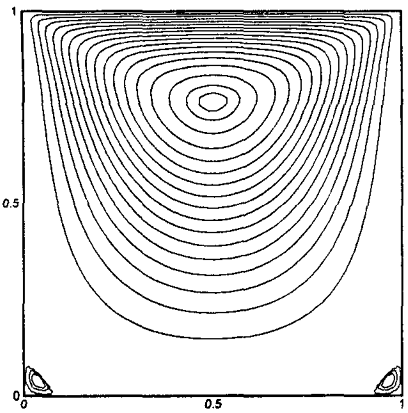
3. Glowinski, R. and Pironneau, O., "Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem," *SIAM Rev.*, Vol.21, 1979, pp.167-212.

4. Tezduyar, T. E., Glowinski, R. and Liou, J., "Petrov-Galerkin methods on multiply connected domains for the vorticity-stream function formulation of the incompressible Navier-Stokes equations," *International Journal for Numerical Methods in Fluids,* Vol.8, 1988, pp.1269-1290.

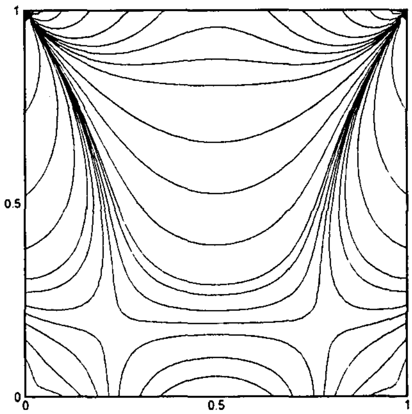
5. Spotz, W. F., "Accuracy and performance of numerical wall boundary conditions for steady, 2S, incompressible stream function vorticity," *International Journal for Numerical Methods in Fluids,* Vol.28, 1998, pp.737-757.

6. Peeters, M. F., Habashi, W. G. and Dueck, E. G., "Finite element streamfunction-vorticity solutions of the incompressible Navier-Stokes equation," *International Journal for Numerical Methods in Fluids,* Vol.7, 1987, pp.17-27.

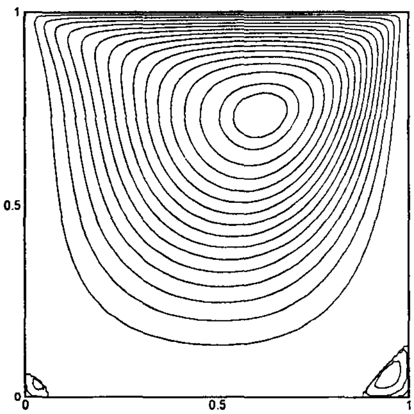
7. Haroutunian, V., Engelman, M. S. and Hasbani, I., "Segregated finite element algorithms for the numerical solution of large-scale incompressible flow problem," *International Journal for Numerical Methods in Fluids,* Vol.17, 1993, pp.323-348.



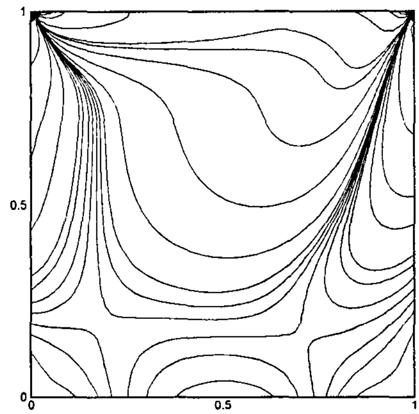
(a) Stream Line : Stokes(Re=0)



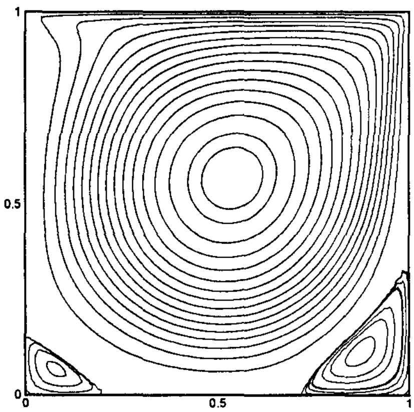
(b) Vorticity : Stokes(Re=0)



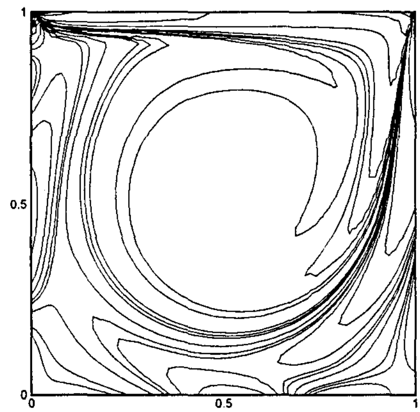
(c) Stream Line : Navier-Stokes(Re=100)



(d) Vorticity : Navier-Stokes(Re=100)



(e) Stream Line : Navier-Stokes(Re=1000)



(f) Vorticity : Navier-Stokes(Re=1000)

Figure 2. Stream lines and Vorticities for Stokes (a, b) and Navier-Stokes (c, d, e, f) driven cavity flows