한국전산구조공학회
2003년도 가을 학술발표회 논문집

Finite Element Formulation using Arbitrary Lagrangian Eulerian Method for Saturated Porous Media

Park, Taehyo* and Jung, Sochan**

ABSTRACT

Porous media consist of physically and chemically different materials and have an extremely complicated behavior due to the different material properties of each of its constituents. In addition, the internal structure of porous media has generally a complex geometry that makes the description of its mechanical behavior quite complex. Thus, in order to describe and clarify the deformation behavior of porous media, constitutive models for deformation of porous media coupling several effects such as flow of fluids of thermodynamical change need to be developed in frame of Arbitrary Lagrangian Eulerian (ALE) description. The aim of ALE formulations is to maximize the advantages of Lagrangian and Eulerian methods, and to minimize the disadvantages. Therefore, this method is appropriate for the analysis of porous media that are considered for the behavior of solids and fluids. First of all, governing equations for saturated porous media based on ALE description are derived. Then, weak forms of these equations are obtained in order to implement numerical method using finite element method. Finally, Petrov-Galerkin method is applied to develop finite element formulation.

1. Introduction

In order to describe the behavior of porous media, there have been many works focusing on consolidation and other phenomena in porous media. The basic equations of motion for porous media in quasi-static state were first established in 1941 by Biot, and derivations with some modifications and numerical schemes for compressible, incompressible fluids are introduced by Zienkiewicz and Shiomi (1984). In the work of Schrefler and Xiaoyong (1993), a fully coupled model is developed to simulate the slow transient phenomena involving flow of water and air in deforming porous media. In addition, there have been a lot of works concerning strain localization phenomenon. For modeling localized material failure, the work by Larsson and Runesson (1996) is proposed to embed the localization band based on regularized discontinuity, and a finite element formulation for capturing strong discontinuities in fluid-saturated porous media is developed by Steinmann (1999).

Although many researches for porous media have been performed so far, the researches have been focused on only solid phase or fluid phase. They seem to have limitations to be exact researches, because porous media have the internal structure and both solids and fluids should be considered together. Therefore Arbitrary Lagrangian Eulerian (ALE) method is appropriate for the analysis of the deformation behavior for porous media and Ale method is used in this work. Since ALE method in

* Member, Assoc. Prof., Dept. of Civil Eng., Hanyang Univ.
** Member, Grad. Student, Dept. of Civil Eng., Hanyang Univ.

continuum mechanics was firstly researched in fluid dynamics problems, ALE method has been widely used for various problems of non-linear solid mechanics by many researchers (Benson, 1989; Ghosh and Kikuchi, 1991). Recently, ALE finite element formulations for finite strain elasticity and plasticity problems are developed by Love (2000).

Governing equations for saturated porous media based on ALE description are obtained. Then, weak forms are represented in order to implement numerical method using finite element method. In addition, Petrov-Galerkin method is necessarily applied to develop finite element formulation, in order to overcome numerical instabilities caused by convective terms in ALE equations.

2. Governing Equations for Saturated Porous Media

Governing equations are obtained from the work of Park and Jung (2003). The mass balance equation of the liquid phase is obtained as follows

$$\left(\frac{\alpha-n}{K^s}+\frac{n}{K^w}\right)\frac{Dp^w}{Dt}+\alpha\mathrm{div}\mathbf{v}^s+\mathrm{div}\left\{\frac{\mathbf{k}}{\mu^w}\left(-\mathrm{grad}\,p^w+\rho^w\left(\mathbf{g}-\mathbf{a}^w\right)\right)\right\}=0 \tag{1}$$

where p^w, ρ^w, $\mathbf{a}^w$, K^w, μ^w is pressure, density, acceleration, bulk modulus, dynamic viscosity of the liquid phase, respectively, α is Biot's constant, and n is porosity. In addition, $\mathbf{v}^s$, K^s is velocity, bulk modulus of the solid phase, respectively, $\mathbf{k}$ is the permeability tensor of the medium, and $\mathbf{g}$ is the external momentum supply related to gravitational effects.

Next, the linear momentum balance equation for the whole multiphase medium is obtained as follows

$$\mathbf{div}\boldsymbol{\sigma}+\rho\mathbf{g}-\rho\mathbf{a}^s=0 \tag{2}$$

where $\boldsymbol{\sigma}$ is total stress tensor acting on a unit area of a multiphase medium, $\mathbf{a}^s$ is the solid phase acceleration, and the averaged density of the multiphase medium ρ is given as follows

$$\rho=(1-n)\rho^s+n\rho^w \tag{3}$$

3. Weak forms

For finite element analysis, governing equations can be written in their weak forms. In addition, the initial and boundary conditions are necessary to be given. The initial conditions of solid displacements $\mathbf{u}$ and liquid pressures p^w in domain of interest Ω and boundary of interest Γ at time $t=0$ are given by

$$\mathbf{u}=\mathbf{u}_0,\ p^w=p_0^w \text{ in } \Omega \text{ and on } \Gamma \tag{4a,b}$$

On the one hand, the imposed boundary conditions for external displacements $\mathbf{u}$ and tractions $\mathbf{t}$ for the displacement field are given as follows.

$$\mathbf{u}=\tilde{\mathbf{u}} \text{ on } \Gamma_u,\ \mathbf{t}=\tilde{\mathbf{t}} \text{ on } \Gamma_u^q \tag{5a,b}$$

where $\Gamma_u\cup\Gamma_u^q=\Gamma$, the symbol tilde "$\tilde{x}$" stands for the imposed value on boundary, and "x"

indicates just an arbitrary character.

On the other hand, the imposed boundary conditions for liquid pressures p^w and fluxes q^w normal to the boundary for the pressure field are given as follows.

$$p^w = \tilde{p}^w \text{ on } \Gamma_w, \quad \rho^w \frac{\mathbf{k}}{\mu^w}\left(-\text{grad}p^w + \rho^w\left(\mathbf{g}-\mathbf{a}^w\right)\right)\cdot\mathbf{n} = q^w \text{ on } \Gamma_w^q \tag{6a,b}$$

where $\Gamma_w \cup \Gamma_w^q = \Gamma$ and $\mathbf{n}$ is the unit vector in the normal direction to the boundary Γ. Since flux means the volume of fluid that passes through a given area per unit time, condition (6) can be from the linear momentum balance equation for each fluid phase (Park and Jung, 2003).

In order to classify weighting functions, the symbol "$\breve{x}$" is used. Applying flux boundary condition (6) and weighting functions $\mathbf{w}$, $\breve{\mathbf{w}}$ into Eq. (1), one obtains as follows

$$\begin{aligned}&\int_\Omega \mathbf{w}\left[\text{div}\left(\frac{\mathbf{k}}{\mu^w}\left(-\text{grad}p^w + \rho^w\left(\mathbf{g}-\mathbf{a}^w\right)\right)\right) + \alpha\text{div}\mathbf{v}^s + \left(\frac{\alpha-n}{K^s}+\frac{n}{K^w}\right)\frac{Dp^w}{Dt}\right]d\Omega \\ &+ \int_{\Gamma_w^q} \breve{\mathbf{w}}\left[\frac{\mathbf{k}}{\mu^w}\left(-\text{grad}p^w + \rho^w\left(\mathbf{g}-\mathbf{a}^w\right)\right)\cdot\mathbf{n} - \frac{q^w}{\rho^w}\right]d\Gamma = 0\end{aligned} \tag{7}$$

It may be assumed that the boundary condition (6) is satisfied by the choice of the approximation of the weighting functions. Since there is liquid flow in the representative volume element of porous media and pressures are applied to the domain Ω, the choice of the weighting functions is limited as follows

$$\mathbf{w} = \mathbf{0} \text{ on } \Gamma_w, \quad \breve{\mathbf{w}} = -\mathbf{w} \text{ on } \Gamma_w^q \tag{8a,b}$$

Green theorem applied to the first term in Eq. (7) and substitution Eq. (8) into Eq. (7) yield the following relation

$$\begin{aligned}&\int_\Omega (\text{div}\mathbf{w})\frac{\mathbf{k}}{\mu^w}\left(\text{grad}p^w\right)d\Omega - \int_\Omega (\text{div}\mathbf{w})\frac{\mathbf{k}}{\mu^w}\rho^w\left(\mathbf{g}-\mathbf{a}^w\right)d\Omega + \int_\Omega \mathbf{w}\left(\alpha\text{div}\mathbf{v}^s\right)d\Omega \\ &+ \int_\Omega \mathbf{w}\left(\frac{\alpha-n}{K^s}+\frac{n}{K^w}\right)\frac{Dp^w}{Dt}d\Omega + \int_{\Gamma_w^q} \mathbf{w}\frac{q^w}{\rho^w}d\Gamma = 0\end{aligned} \tag{9}$$

When the ALE method is enforced to Eq. (9), $\mathbf{a}^w$ in the second term and $\frac{Dp^w}{Dt}$ in the fourth term in Eq. (9) are obtained as follows

$$\mathbf{a}^w = \frac{D\mathbf{v}^w}{Dt} = \frac{\partial \mathbf{v}^w(\chi,t)}{\partial t} + \text{grad}\mathbf{v}^w\cdot\mathbf{c} = \mathbf{v}^w_{,t[\chi]} + \text{grad}\mathbf{v}^w\cdot\mathbf{c} \tag{10}$$

$$\frac{Dp^w}{Dt} = \frac{\partial p^w(\chi,t)}{\partial t} + \text{grad}p^w\cdot\mathbf{c} = p^w_{,t[\chi]} + \text{grad}p^w\cdot\mathbf{c} \tag{11}$$

where $\mathbf{c}$ is the convective velocity vector and is the difference between the material velocity vector and the mesh velocity vector.

Therefore, introduction Eqs. (10) and (11) into Eq. (9) yields the following relation. Eq. (12) is the weak form for mass balance equation for the liquid phase.

$$\int_\Omega (\text{div}\mathbf{w})\frac{\mathbf{k}}{\mu^w}(\text{grad}p^w)d\Omega - \int_\Omega (\text{div}\mathbf{w})\frac{\mathbf{k}}{\mu^w}\rho^w\mathbf{g}d\Omega + \int_\Omega (\text{div}\mathbf{w})\frac{\mathbf{k}}{\mu^w}\rho^w\left(\mathbf{v}^w_{,t[x]} + \text{grad}\mathbf{v}^w\cdot\mathbf{c}\right)d\Omega$$
$$+\int_\Omega \mathbf{w}\left(\alpha\text{div}\mathbf{v}^s\right)d\Omega + \int_\Omega \mathbf{w}\left(\frac{\alpha-n}{K^s}+\frac{n}{K^w}\right)\left(p^w_{,t[x]} + \text{grad}p^w\cdot\mathbf{c}\right)d\Omega + \int_{\Gamma^q_w}\mathbf{w}\frac{q^w}{\rho^w}d\Gamma = 0 \tag{12}$$

In order to represent other weighting functions, the symbol "$*$" and "$\breve{x}$" are used. Applying traction boundary condition (5) and other weighting functions $\mathbf{w}^*$, $\breve{\mathbf{w}}^*$ into Eq. (2), one obtains as follows

$$\int_\Omega \mathbf{w}^*(\text{div}\boldsymbol{\sigma} + \rho\mathbf{g})d\Omega - \int_\Omega \mathbf{w}^*\rho\mathbf{a}^s d\Omega + \int_{\Gamma^q_u}\breve{\mathbf{w}}^*\left(\mathbf{t}-\tilde{\mathbf{t}}\right)d\Gamma = 0 \tag{13}$$

It may be assumed that the boundary condition (5) is satisfied by the choice of the approximation of the weighting functions. Since there are solid deformation and liquid flow in the representative volume element of porous media and displacements are related to the domain Ω, the choice of the weighting functions is limited as follows

$$\mathbf{w}^* = \mathbf{0} \text{ on } \Gamma_u, \quad \breve{\mathbf{w}}^* = -\mathbf{w}^* \text{ on } \Gamma^q_u \tag{14a,b}$$

Green theorem applied to the first term in Eq. (13) and substitution Eq. (14) into (13) yield the following relation

$$-\int_\Omega \left(\text{div}\mathbf{w}^*\right)\boldsymbol{\sigma}d\Omega + \int_\Omega \mathbf{w}^*\rho\mathbf{g}d\Omega - \int_\Omega \mathbf{w}^*\rho\mathbf{a}^s d\Omega + \int_{\Gamma^q_u}\mathbf{w}^*\tilde{\mathbf{t}}d\Gamma = 0 \tag{15}$$

When the ALE method is enforced to Eq. (15), $\mathbf{a}^s$ in the third term in Eq. (15) is obtained as follows

$$\mathbf{a}^s = \frac{D\mathbf{v}^s}{Dt} = \frac{\partial\mathbf{v}^s(\chi,t)}{\partial t} + \text{grad}\mathbf{v}^s\cdot\mathbf{c} = \mathbf{v}^s_{,t[x]} + \text{grad}\mathbf{v}^s\cdot\mathbf{c} \tag{16}$$

Finally, introduction Eq. (16) into Eq. (15) yields the following relation. Eq. (17) is the weak form for linear momentum equation for the whole multiphase medium

$$\int_\Omega \left(\text{div}\mathbf{w}^*\right)\boldsymbol{\sigma}d\Omega = \int_\Omega \mathbf{w}^*\rho\mathbf{g}d\Omega - \int_\Omega \mathbf{w}^*\rho\left(\mathbf{v}^s_{,t[x]} + \text{grad}\mathbf{v}^s\cdot\mathbf{c}\right)\mathbf{a}^s d\Omega + \int_{\Gamma^q_u}\mathbf{w}^*\tilde{\mathbf{t}}d\Gamma \tag{17}$$

4. Finite Element Formulation

The following equations are represented together with vector notation and matrix symbolic notation. Voigt notation is introduced from following relations, which means any other conversion of symmetric higher order tensors to column matrices. Column matrices are represented as brace notation $\{\ \}$ denoted by letters enclosed within braces, In addition, row matrices or rectangular matrices are represented as bracket notation $[\ ]$ denoted by letters enclosed within brackets.

The fourth term in Eq. (12) can be obtained as follows

$$\mathrm{div}\{\mathbf{v}^s\} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial u_3}{\partial t} \end{Bmatrix} = \{\mathbf{I}_0\}^T [\partial] \frac{\partial \{\mathbf{u}\}}{\partial t} \tag{18}$$

where

$$\{\mathbf{I}_0\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad [\partial] = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix} \tag{19a,b}$$

Substituting Eq. (18) into Eq. (12) and using differential operator $[\partial]$, gradient operator ∇, divergence operator ∇^T, Eq. (12) is obtained as follows

$$\begin{aligned} &\int_\Omega [\mathbf{w}]^T \nabla^T \frac{[\mathbf{k}]}{\mu^w} (\nabla p^w) d\Omega - \int_\Omega [\mathbf{w}]^T \nabla^T \frac{[\mathbf{k}]}{\mu^w} \rho^w \{\mathbf{g}\} d\Omega + \int_\Omega [\mathbf{w}]^T \nabla^T \frac{[\mathbf{k}]}{\mu^w} \rho^w \left(\{\mathbf{v}^w_{,t[x]}\} + \nabla \{\mathbf{v}^w\} \cdot \{\mathbf{c}\} \right) d\Omega \\ &+ \int_\Omega [\mathbf{w}]^T \alpha \{\mathbf{I}_0\}^T [\partial] \frac{\partial \{\mathbf{u}\}}{\partial t} d\Omega + \int_\Omega [\mathbf{w}]^T \left(\frac{\alpha - n}{K^s} + \frac{n}{K^w} \right) \left(p^w_{,t[x]} + \nabla p^w \cdot \{\mathbf{c}\} \right) d\Omega + \int_{\Gamma^q_w} [\mathbf{w}]^T \frac{q^w}{\rho^w} d\Gamma = 0 \end{aligned} \tag{20}$$

where

$$\nabla^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix} \tag{21}$$

Moreover, using differential operator matrix $[\partial]$, Eq. (17) is obtained as follows

$$\int_\Omega [\mathbf{w}^*]^T [\partial]^T \{\boldsymbol{\sigma}\} d\Omega = \int_\Omega [\mathbf{w}^*]^T \rho \{\mathbf{g}\} d\Omega - \int_\Omega [\mathbf{w}^*]^T \rho \left(\{\mathbf{v}^s_{,t[x]}\} + \nabla \{\mathbf{v}^s\} \cdot \{\mathbf{c}\} \right) d\Omega + \int_{\Gamma^q_u} \{\mathbf{w}^*\}^T \{\tilde{\mathbf{t}}\} d\Gamma \tag{22}$$

When Galerkin finite element formulation is applied to equations in ALE description, numerical instabilities are encountered because of convective terms in material time derivatives. In order to overcome such numerical instabilities, Petrov-Galerkin finite element formulation should be introduced. In Petrov-Galerkin formulation, the trial and test shape functions differ (Belytschko et al., 2000).

Displacements $\{\mathbf{u}\}$ and pore pressure p^w as unknown field variables are interpolated with nodal values, and convective velocity $\{\mathbf{c}\}$ can be obtained as follows

$$\{\mathbf{u}\}=[\mathbf{N}^u]\{\mathbf{u}^u\}\ ,\ \mathbf{u}(\chi(\boldsymbol{\xi}),t)=\mathbf{u}_I^u(t)N_I^u(\boldsymbol{\xi}) \tag{23a,b}$$

$$p^w=[\mathbf{N}^p]\{\mathbf{p}^w\}\ ,\ p^w(\chi(\boldsymbol{\xi}),t)=p_I^w(t)N_I^p(\boldsymbol{\xi}) \tag{24a,b}$$

$$\{\mathbf{c}\}=[\mathbf{N}](\{\mathbf{v}\}-\{\hat{\mathbf{v}}\})\ ,\ \mathbf{c}(\chi(\boldsymbol{\xi}),t)=\mathbf{c}_I(t)N_I(\boldsymbol{\xi})=[\mathbf{v}_I(t)-\hat{\mathbf{v}}_I(t)]N_I(\boldsymbol{\xi}) \tag{25a,b}$$

where I denotes I th node.

In order to classify shape functions, the symbol bar "$\overline{x}$" is used. Test shape functions for the pore pressure and displacement are denoted by $[\overline{\mathbf{N}}^p]$ and $[\overline{\mathbf{N}}^u]$, respectively.

$$[\overline{\mathbf{N}}^p]=[\mathbf{N}^p]+\tau\{\mathbf{c}\}\cdot\nabla[\mathbf{N}^p],\ \overline{N}_I^p=N_I^p+\tau c_j N_{I,j}^p \tag{26a,b}$$

$$[\overline{\mathbf{N}}^u]=[\mathbf{N}^u]+\tau\{\mathbf{c}\}\cdot\nabla[\mathbf{N}^u],\ \overline{N}_I^u=N_I^u+\tau c_j N_{I,j}^u \tag{27a,b}$$

where τ denotes stabilization parameter, and j indicates the tensorial index.

The material time derivative of pore pressure p^w in ALE description can be interpolated as follows

$$\frac{Dp^w}{Dt}=p^w_{,t[x]}+\{\mathbf{c}\}\cdot\nabla p^w=\frac{\partial p_I^w(t)}{\partial t}N_I^p(\boldsymbol{\xi})+\mathbf{c}(\boldsymbol{\xi},t)\cdot\nabla N_I^p(\boldsymbol{\xi})p_I^w(t) \tag{28}$$

The third term $\nabla\{\mathbf{v}^w\}$ in Eq. (20) and the second term $\nabla\{\mathbf{v}^s\}$ in Eq. (22) can vanish because of slow deformation and flow from basic assumptions. In addition, if high-frequency dynamic phenomena are not considered, $\frac{\partial\{\mathbf{v}^w\}}{\partial t}$ and $\frac{\partial\{\mathbf{v}^s\}}{\partial t}$ can be equal to $\frac{\partial^2\{\mathbf{u}\}}{\partial t^2}$ (Zienkiewicz et al., 1990).

In Eq. (20), substitution $[\mathbf{w}]$ into $[\overline{\mathbf{N}}^p]$ as test shape function, and introduction of $[\mathbf{N}^p]$ and $[\mathbf{N}^u]$ as trial shape functions yield the following relation

$$\begin{aligned}&\int_\Omega(\nabla[\overline{\mathbf{N}}^p])^T\frac{[\mathbf{k}]}{\mu^w}\nabla[\mathbf{N}^p]\{\mathbf{p}^w\}d\Omega-\int_\Omega(\nabla[\overline{\mathbf{N}}^p])^T\frac{[\mathbf{k}]}{\mu^w}\rho^w\{\mathbf{g}\}d\Omega\\&+\int_\Omega(\nabla[\overline{\mathbf{N}}^p])^T\frac{[\mathbf{k}]}{\mu^w}\rho^w[\mathbf{N}^u]\frac{\partial^2\{\mathbf{u}^u\}}{\partial t^2}d\Omega+\int_\Omega[\overline{\mathbf{N}}^p]^T\alpha\{\mathbf{I}_0\}^T[\partial][\mathbf{N}^u]\frac{\partial\{\mathbf{u}^u\}}{\partial t}d\Omega\\&+\int_\Omega[\overline{\mathbf{N}}^p]^T\left(\frac{\alpha-n}{K^s}+\frac{n}{K^w}\right)\left(\frac{\partial\{\mathbf{p}^w\}}{\partial t}[\mathbf{N}^p]+\{\mathbf{c}\}\cdot\nabla[\mathbf{N}^p]\{\mathbf{p}^w\}\right)d\Omega+\int_{\Gamma_w^q}[\overline{\mathbf{N}}^p]^T\frac{q^w}{\rho^w}d\Gamma=0\end{aligned} \tag{29}$$

In Eq. (22), substitution $[\mathbf{w}^*]$ into $[\overline{\mathbf{N}}^u]$ as test shape function and introduction of $[\mathbf{N}^u]$ as trial shape function yield the following relation

$$\int_\Omega([\partial][\overline{\mathbf{N}}^u])^T\{\sigma\}d\Omega=\int_\Omega[\overline{\mathbf{N}}^u]^T\rho\{\mathbf{g}\}d\Omega-\int_\Omega[\overline{\mathbf{N}}^u]^T\rho[\mathbf{N}^u]\frac{\partial^2\{\mathbf{u}^u\}}{\partial t^2}d\Omega+\int_{\Gamma_u^q}[\overline{\mathbf{N}}^u]^T\{\tilde{\mathbf{t}}\}d\Gamma \tag{30}$$

Eq. (29) can be represented with matrix symbolic forms as follows and Eq. (31) is finite element formulation for mass balance for the liquid phase.

$$[\mathbf{H}]\{\mathbf{p}^w\}+[\mathbf{C}]\{\mathbf{p}^w\}+[\mathbf{Q}]\frac{\partial\{\mathbf{u}^u\}}{\partial t}+[\mathbf{G}]\frac{\partial^2\{\mathbf{u}^u\}}{\partial t^2}+[\mathbf{S}]\frac{\partial\{\mathbf{p}^w\}}{\partial t}=\{\mathbf{f}^p\} \tag{31}$$

where the permeability matrix $[\mathbf{H}]$, the convective velocity matrix $[\mathbf{C}]$, the first connecting matrix $[\mathbf{Q}]$, the seepage matrix $[\mathbf{G}]$, the compressibility matrix $[\mathbf{S}]$, and the flow vector $\{\mathbf{f}^p\}$ are respectively shown as follows

$$[\mathbf{H}]=\int_\Omega(\nabla[\overline{\mathbf{N}}^p])^T\frac{[\mathbf{k}]}{\mu^w}\nabla[\mathbf{N}^p]d\Omega\ ,\ [\mathbf{C}]=\int_\Omega[\overline{\mathbf{N}}^p]^T\left(\frac{\alpha-n}{K^s}+\frac{n}{K^w}\right)\{\mathbf{c}\}\cdot\nabla[\mathbf{N}^p]d\Omega \tag{32a,b}$$

$$[\mathbf{Q}]=\int_\Omega[\overline{\mathbf{N}}^p]^T\alpha\{\mathbf{I}_0\}^T[\partial][\mathbf{N}^u]d\Omega\ ,\ [\mathbf{G}]=\int_\Omega(\nabla[\overline{\mathbf{N}}^p])^T\frac{[\mathbf{k}]}{\mu^w}\rho^w[\mathbf{N}^u]d\Omega \tag{33a,b}$$

$$[\mathbf{S}]=\int_\Omega[\overline{\mathbf{N}}^p]^T\left(\frac{\alpha-n}{K^s}+\frac{n}{K^w}\right)[\mathbf{N}^p]d\Omega\ ,\ \{\mathbf{f}^p\}=\int_\Omega(\nabla[\overline{\mathbf{N}}^p])^T\frac{[\mathbf{k}]}{\mu^w}\rho^w\{\mathbf{g}\}d\Omega-\int_{\Gamma_w^q}[\overline{\mathbf{N}}^p]^T\frac{q^w}{\rho^w}d\Gamma \tag{34a,b}$$

In Eq. (30), the total stress $\{\boldsymbol{\sigma}\}$ can be equal to $\{\boldsymbol{\sigma}''\}-\alpha\{\mathbf{I}_0\}p^w$ using the effective stress $\{\boldsymbol{\sigma}''\}$ (Zienkiewicz and Shiomi, 1984), and one obtains as follows

$$\begin{aligned}&\int_\Omega([\partial][\overline{\mathbf{N}}^u])^T\{\boldsymbol{\sigma}''\}d\Omega-\int_\Omega([\partial][\overline{\mathbf{N}}^u])^T\alpha\{\mathbf{I}_0\}([\mathbf{N}^p]\{\mathbf{p}^w\})d\Omega\\&=\int_\Omega[\overline{\mathbf{N}}^u]^T\rho\{\mathbf{g}\}d\Omega-\int_\Omega[\overline{\mathbf{N}}^u]^T\rho[\mathbf{N}^u]\frac{\partial^2\{\mathbf{u}^u\}}{\partial t^2}d\Omega+\int_{\Gamma_u^q}[\overline{\mathbf{N}}^u]^T\{\tilde{\mathbf{t}}\}d\Gamma\end{aligned} \tag{35}$$

Eq. (35) can be represented with matrix symbolic forms as follows and Eq. (36) is finite element formulation for linear momentum balance for the whole mixture.

$$\int_\Omega[\mathbf{B}]\{\boldsymbol{\sigma}''\}d\Omega-[\mathbf{R}]\{\mathbf{p}^w\}+[\mathbf{M}]\frac{\partial\{\mathbf{u}^u\}}{\partial t}=\{\mathbf{f}^u\} \tag{36}$$

where the differential shape function matrix $[\mathbf{B}]$, the second connecting matrix $[\mathbf{R}]$, the mass matrix $[\mathbf{M}]$, and the external load vector $\{\mathbf{f}^u\}$ are respectively shown as follows

$$[\mathbf{B}]=([\partial][\overline{\mathbf{N}}^u])^T\ ,\ [\mathbf{R}]=\int_\Omega[\mathbf{B}]\alpha\{\mathbf{I}_0\}[\mathbf{N}^p]d\Omega \tag{37a,b}$$

$$[\mathbf{M}]=\int_\Omega[\overline{\mathbf{N}}^u]^T\rho[\mathbf{N}^u]d\Omega\ ,\ \{\mathbf{f}^u\}=\int_\Omega[\overline{\mathbf{N}}^u]^T\rho\{\mathbf{g}\}d\Omega+\int_{\Gamma_u^q}[\overline{\mathbf{N}}^u]^T\{\tilde{\mathbf{t}}\}d\Gamma \tag{38a,b}$$

5. Conclusion

ALE method is used to maximize the advantage and minimize the disadvantage of Lagrangian and Eulerian elements in Lagrangian and Eulerian methods. In the study of porous media, the internal structure should be considered, which consists of a solid phase as well as closed and open pores. Since the internal structure has materials such as solids and fluids moving with a relative velocity, ALE method is appropriate for the analysis of the deformation behavior for porous media. For this reason, governing equations for saturated porous media with some assumptions are obtained with the frame of ALE description in this work. Weak forms are represented in order to implement numerical method using finite element method and Petrov-Galerkin method is necessarily applied to develop finite element formulation, because of avoiding numerical instabilities caused by convective terms in ALE equations. After numerical implementation, the numerical results will be verified with existing practical data in the forthcoming paper.

Acknowledgements

This work was supported by grant No. R01-2002-000-00063-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

References

Belytschko, T., Liu, W.K. and Moran, B., *Nonlinear Finite Elements for Continua and Structures*, John Wiley & Sons, Chichester, 2000, p.650

Benson, D.J., "An Efficient Accurate, Simple ALE Method for Nonlinear Finite Element Programs," *Computer Methods in Applied Mechanics and Engineering*, Vol.72, 1989, pp.305~350

Biot, M.A., "General Theory of Three-Dimensional Consolidation," *Journal of Applied Physics*, Vol.12, 1941, pp.155~164

Ghosh, S. and Kikuchi, N., "An Arbitrary Lagrangian-Eulerian Finite Element Method for Large Deformation Analysis of Elastic-Viscoplastic Solids," *Computer Methods in Applied Mechanics and Engineering*, Vol.86, 1991, pp.127~188

Larsson, R. and Runesson, K., "Element-Embedded Localization Band Based on Regularized Displacement Discontinuity," *Journal of Engineering Mechanics*, Vol.122, 1996, pp.402~411

Lewis, R.W. and Schrefler, B.A., *The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media*, John Wiley & Sons, Chichester, 1998, p.492

Love, E., *Arbitrary* Lagrangian*-Eulerian (ALE) Finite Element Formulations in Finite Strain Elasto-Plasticity*, Ph.D. Dissertation, University of California, Berkeley, 2000, p.266

Park, T. and Jung, S., "Theoretical Formulation for Behavior of Saturated Porous Media using Arbitrary Lagrangian Eulerian Method," *KSCE Journal of Civil Engineering*, 2003, submitted for publication.

Schrefler, B.A. and Xiaoyong, Z., "A Fully Coupled Model for Water Flow and Airflow in Deformable Porous Media," *Water Resources Research*, Vol.29, No.1, 1993, pp.155~167

Steinmann, P., "A Finite Element Formulation for Strong Discontinuities in Fluid-saturated Porous Media," *Mechanics of Cohesive-Frictional Materials*, Vol.4, 1999, pp.133~152

Zienkiewicz, O.C., Chan, A.H., Pastor, M., Paul, D.K. and Shiomi, T., "Static and Dynamic Behaviour of soils: A Rational Approach to Quantitative Solution, I. Fully Saturated Problems," *Proceedings of the Royal Society of London. Series A*, Vol.429, 1990, pp.285~309

Zienkiewicz, O.C. and Shiomi, T., "Dynamic Behaviour of Saturated Porous Media: The General Biot Formulation and Its Numerical Solution," *International Journal for Numerical and Analytical Methods in Geomechanics*, Vol.8, 1984, pp.71~96