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Free Vibrations of Horizontally Curved Beams with General Boundary Condition

일반경계 조건을 갖는 수평 곡선보의 자유진동

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Abstract

This paper deals with the free vibrations of horizontally curved beams with the general boundary condition, which consists of translational and rotational springs. The equations of general boundary condition of such beams are derived, while the ordinary differential equations governing free vibrations are adopted from the literature. The parabola as the curved beam's curvilinear shape is considered in numerical examples. For calculating the natural frequencies, the governing equations are solved by numerical methods. The Runge-Kutta and Determinant Search Methods are used for integrating the differential equations and for calculating the natural frequencies, respectively. For validation purpose, the numerical results obtained herein are compared to those obtained from the SAP 2000. With regard to numerical results, the relationships between frequency parameters and various beam parameters are presented in the forms of Table and Figures.

Keywords: free vibration, curved beams, general boundary condition, natural frequency, rotational spring, translational spring.

1. Introduction

Studies on the free vibrations of linearly elastic, horizontally curved beams of various shapes have been reported for more than three decades. These studies were critically reviewed by Lee, *et al.*(1) Briefly, such works included studies of circular curved beams with predictions of the lowest frequency by Volterra and Morell(2), Romanelli and Laura(3), and Maurizi, *et al.*(4); studies of non-circular curved beams with predictions of higher frequencies by Shore and Chaudhuri(5), Kawakami, *et al.*(6), Kang, *et al.*(7), Yildirim (8), and Lee, *et al.*(1); and studies showing the effect of rotatory inertia on free vibration frequencies by Rao(9), Laskey(10), and Mo(11).

All the works mentioned above include the curved beams with only perfect end constraints, while works on general boundary condition should not be seen in the open literature.

Concerning with the straight beams with the general boundary condition, Li(12) and Lee *et al.*(13) studied the free vibrations of such members with the general boundary condition. Especially, only the uniform beams have been considered in the former, while not only uniform beams but also linearly tapered beams have been taken into account in the latter for the parametric studies of beams.

In real structural system, the actual supports are never perfectly rigid, and so, for instance, there will always be a small amount of translation against immovability and a small amount of restraint against free rotation at a pin support(14). These deviations from idealized conditions cannot be disregarded and thus should be taken into account for the analyses of structures in general and for the dynamic problems of structures in particular. Such imperfect supports should be modeled as the general boundary condition, which consists of two elastic translational and rotational springs(12).

In view of the above, the purpose of this paper is to investigate the characteristics of free vibrations of horizontally curved beams with the general boundary condition.

The following assumptions are inherent in this theory: the beam is linearly elastic; the effects of rotatory inertia and shear deformation are considered; and the small deflection theory is governed. In addition, the beam is assumed to be in harmonic motion.

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2. Mathematical Model

The geometry of horizontally curved beam placed on the rectangular co-ordinates (x, y), symmetric about the crown, is depicted in Fig. 1. Its span length, horizontal rise and shape of the middle surface are l, h, and $y(x)$, respectively. Its radius of curvature ρ, a function of the co-ordinate x, has an inclination θ with the radius of left support. Each end of the beam is supported by two separate elastic springs, i.e. the translational and rotational springs, as shown in Fig. 1. Its translational spring constants of left and right ends are K_{va} and K_{vb}, and rotational spring constants are K_{ba} and K_{bb}, respectively. However it is assumed that the torsional angle at each end is not allowed as shown in the Fig. 1. The beam has a uniform cross-section with the area A, the area moment of inertia of the cross-section I, and the polar moment of inertia of the cross-section J. Also, depicted in Fig. 1 are positive directions of the vertical displacement v, the rotation due to bending ψ, rotation due to shear β, and the torsional angle ϕ at any coordinates (x, y).

The beam is assumed to be in harmonic motion, or each co-ordinate is proportional to $\sin(\omega_i t)$, such as

$$v(x,t) = v\sin(\omega_i t) \tag{1}$$

$$\psi(x,t) = \psi\sin(\omega_i t) \tag{2}$$

$$\phi(x,t) = \phi\sin(\omega_i t) \tag{3}$$

where ω_i is the angular frequency, t is time and i is mode number.

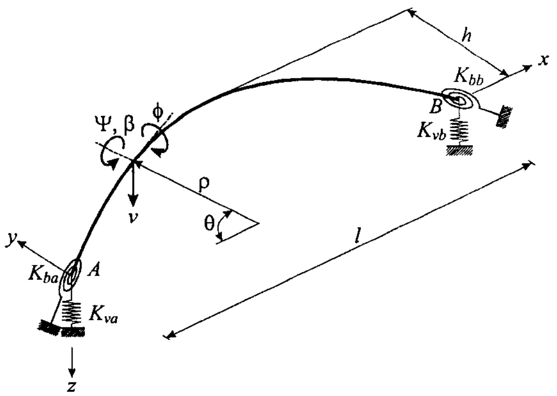

Fig. 1 Horizontally Curved Beam with General Boundary Condition and Its Variables

To facilitate the numerical studies and to obtain the most general results for this class of problem, the following non-dimensional system variables are defined. The horizontal rise to span length ratio f, slenderness ratios s and s_p, shear parameter μ, and stiffness parameter e, respectively, are

$$f = h / l \tag{4}$$

$$s = l / (I / A)^{1/2} \tag{5}$$

$$s_p = l / (J / A)^{1/2} \tag{6}$$

$$\mu = \kappa G / E \tag{7}$$

$$e = GJ / (EI) \tag{8}$$

where κ is the shape factor of the cross-section, and E and G are Young's modulus and shear modulus, respectively.

The co-ordinates, displacement and the radius of curvature are normalized by span length l:

$$\xi = x / l \tag{9}$$

$$\eta = y / l \tag{10}$$

$$\delta = v / l \tag{11}$$

$$\zeta = \rho / l \tag{12}$$

Finally, the frequency parameter is defined as

$$c_i = \omega_i l^2 (\gamma A / EI)^{1/2}, \quad i = 1, 2, 3, 4, \Lambda \tag{13}$$

where γ is the mass density.

Using both the stress resultants and the inertia forces and Eqs. (1) through (13) give the following non-dimensional differential equations governing free vibrations of horizontally curved beams with the variable curvature[11], that include the effects of rotatory inertia and shear deformation. Note that in this study, the governing equations are not derived but adopted from the literature.

$$\delta''' = a_1 \delta' + c_i^2 a_2 \delta + a_3 \psi' \tag{14}$$

$$\psi'' = a_4 \delta' + a_1 \psi' + (e + a_5 + c_i^2 a_6)\psi + (1+e)\phi' + a_7 \phi \tag{15}$$

$$\phi'' = -(1 + e^{-1})\psi' + a_1 \psi + a_1 \phi' + e^{-1}(1 + c_i^2 a_8)\phi \tag{16}$$

where $(') = d / d\phi$, and the coefficients of a_1 through a_8 are

$$a_1 = \zeta' \zeta^{-1} \tag{17.1}$$

$$a_2 = -\mu^{-1}s^{-2}\zeta^2 \quad (17.2)$$
$$a_3 = \zeta \quad (17.3)$$
$$a_4 = -\mu s^2\zeta \quad (17.4)$$
$$a_5 = \mu s^2\zeta^2 \quad (17.5)$$
$$a_6 = -s^{-2}\zeta^2 \quad (17.6)$$
$$a_7 = -\zeta'\zeta^{-1} \quad (17.7)$$
$$a_8 = -s_p^{-2}\zeta^2 \quad (17.8)$$

3. Boundary Conditions

Now consider the boundary conditions. The shear force Q and bending moment M can be expressed in terms of amplitude v, ψ and ϕ are given as follows [11].

$$Q = \kappa AG\beta = \kappa AG(\rho^{-1}v' - \psi) \quad (18)$$
$$M = EI\rho^{-1}(\phi - \psi') \quad (19)$$

Each end of beam is supported by two separate springs, i.e. the linearly elastic translational and rotational springs, as shown in Fig. 1. At left end ($x = 0$), the spring constants are K_{va} and K_{ba}. Therefore, the shear force Q and the bending moment M corresponding to K_{va} and K_{ba} with deflection v and rotation ψ can be expressed as follows.

$$Q = -K_{va}v \quad (20)$$
$$M = K_{ba}\psi \quad (21)$$

In addition, to facilitate the numerical studies, the boundary conditions are cast in the following non-dimensional forms of spring parameters.

$$k_{\delta a} = K_{va}l^3/(\pi^4 EI) \quad (22)$$
$$k_{ba} = K_{ba}l/(\pi^2 EI) \quad (23)$$
$$k_{\delta b} = K_{va}l^3/(\pi^4 EI) \quad (24)$$
$$k_{bb} = K_{bb}l/(\pi^2 EI) \quad (25)$$

Using Eqs. (18) and (20) combined with Eqs. (11), (12) and (22) give the non-dimensional boundary equation at the left end as follows.

$$\zeta^{-1}\delta' - \psi + \pi^4\mu^{-1}s^{-2}k_{\delta a}\delta = 0 \quad (26)$$

Also, combining Eqs. (12) and (23), and using Eqs. (19) and (21) give the non-dimensional boundary equation as follows.

$$\phi - \psi' - \pi^2 k_{ba}\zeta\psi = 0 \quad (27)$$

At right end ($x = l$), the spring constants are K_{vb} and K_{bb}, and thus, Q and M are expressed by

$$Q = K_{vb}v \quad (28)$$
$$M = -K_{bb}\psi \quad (29)$$

Similarly, combined with Eqs. (11), (12), (23) and (25) using Eqs. (18) and (20), and Eqs. (19) and (21) give the non-dimensional boundary equations at the right end as follows.

$$\zeta^{-1}\delta' - \psi - \pi^4\mu^{-1}s^{-2}k_{\delta b}\delta = 0 \quad (30)$$
$$\phi - \psi' + \pi^2 k_{bb}\zeta\psi = 0 \quad (31)$$

Finally, the torsional angles are not allowed at both left and right ends ($x = 0$ and $x = l$) and the non-dimensional boundary condition is given as follows.

$$\phi = 0 \quad (32)$$

4. Shape Function

The coefficients defined by Eqs. (4) through (8) are computed as follows. Cast the given arch shape $y = y(x)$ in non-dimensional form using Eqs. (9) and (10). This leads to

$$\eta = \eta(\xi) \quad (33)$$

Both θ and ζ are computed from the Eq. (33) expressed as the function of single variable ξ. By mathematical definition,

$$\theta = \pi/2 - \tan(\eta^i) \quad (34)$$
$$\zeta = (1 + \eta^{i2})^{3/2}/\eta^{ii} \quad (35)$$

where $(i) = d/d\xi$.

Then ζ' is calculated from the derivatives of Eqs. (34) and (35) by using

$$\zeta' = (d\zeta/d\xi)(d\xi/d\theta) \quad (36)$$

Consider now the shape function of beam geometry. In this study, the parabolic beam is considered as the beam geometry. The general equation for the parabolic beam of span length l and horizontal rise h is

$$y = -(4h/l^2)x(x-l), \; 0 \le x \le l \quad (38)$$

With Eqs. (4), (9) and (10), the non-dimensional form of Eq. (38) becomes

$$\eta = -4f\xi(\xi-1), \; 0 \le \xi \le 1 \quad (39)$$

With Eq. (39), the following quantity is calculated from Eq. (34):

$$\theta = \pi/2 - \tan^{-1}[-4f(2\xi-1)] \quad (40)$$

Also, the terms of radius of curvature ζ and its derivatives ζ' are calculated by using Eqs. (35), and (36) with Eq. (39). That is,

$$\zeta = 0.125f^{-1}[1+16f^2(2\xi-1)^2]^{3/2} \quad (42)$$

$$\zeta' = 1.5(2\xi-1)[1+16f^2(2\xi-1)^2]^{3/2} \quad (43)$$

Thus, with the coefficient a_1 through a_8 expressed in terms of the single variable ξ and with the end conditions of Eqs. (26), (27), (30), (31) and (32), Eqs. (14), (15) and (16) can be solved numerically to determine the frequency parameters c_i and the corresponding mode shapes $\delta = \delta(\theta)$, $\psi = \psi(\theta)$ and $\phi = \phi(\theta)$ for parabolic beam with the general boundary condition.

5. Numerical Methods and Discussion

Based on the above theory, a computer program has been written to calculate frequency parameters c_i and corresponding mode shapes $\delta = \delta_i(\theta)$, $\psi = \psi_i(\theta)$ and $\phi = \phi_i(\theta)$, but not shown in this paper. The numerical methods described by Lee and Wilson[15], and Lee *et al.*[1] were used to solve the differential Eqs. (14), (15) and (16) subjected to the boundary conditions of Eqs. (26), (27), (30), (31) and (32). Firstly, the Determinant Search method combined with the Regula-Falsi method[16] was used to obtain the frequency parameter, c_i, and

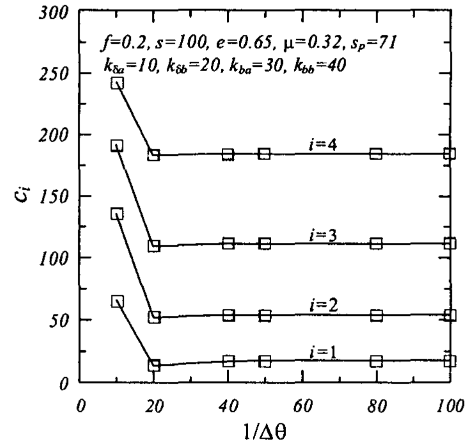

Fig. 2 Convergence Analysis

then the Runge-Kutta method[16] was used to calculate the mode shapes δ and λ.

Prior to executing the numerical study, the convergence analysis, for which $f=0.2$, $s=100$, $e=0.65$, $\mu=0.32$, $s_p=71$, $k_{\delta a}=10$, $k_{\delta b}=20$, $k_{ba}=30$ and $k_{bb}=40$, was conducted to determine appropriate step size $\Delta\theta$ in the Runge-Kutta method. Figure 2 shows $1/\Delta\theta$ versus c_i curves, in which a step size of $\Delta\theta = 1/50$ is found to give convergence for c_i to within three significant figures.

Four lowest values of $c_i (i=1,2,3,4)$ were calculated in this study. Numerical results, given in Table 1 and Figs. 3 through 6, are now discussed hereafter. The first series of numerical results are shown in Table 1. These studies served as preliminary check on the analysis presented herein. For comparative purposes, finite element solutions based on the commercial packages SAP 2000 were used to compute the first four frequency parameters c_i for the two geometry cases. The results showed

Table 1 Comparisons of c_i between this study and SAP 2000 for $e=0.65$ $\mu=0.32$, and $s_p=0.71s$

Geometry	i	Frq. parameter, c_i		A/B
		This study(A)	SAP 2000(B)	
$f=0.2, s=100$, $k_{\delta a}=10, k_{\delta b}=20$, $k_{ba}=30, k_{bb}=40$	1	17.75	17.79	0.998
	2	54.38	53.78	1.011
	3	111.8	108.6	1.029
	4	184.3	184.9	0.997
$f=0.3, s=50$, $k_{\delta a}=k_{\delta b}=10$, $k_{ba}=k_{bb}=20$	1	13.50	13.30	1.015
	2	41.93	42.61	0.984
	3	85.48	86.08	0.993
	4	101.2	102.7	0.985

that 100 three- dimensional finite frame elements were necessary to match within a tolerance of about 3% values of c_i computed by solving the governing differential equations Eqs. (14), (15) and (16) in this study. It can be concluded that the present study gives accurate results.

It is shown in Fig. 3, for which $s=100$, $k_{\delta a}=10$, $k_{\delta b}=20$, $k_{ba}=30$ and $k_{bb}=40$, that each frequency curve except 4th mode is decreased as the rise to span length ratio f is increased. However, the 4th frequency curves reaches a peak as f is increased.

It is shown in Fig. 4, for which $f=0.2$, $k_{\delta a}=10$, $k_{\delta b}=20$, $k_{ba}=30$ and $k_{bb}=40$, that the frequency parameters are initially increased, then in most cases approach a horizontal asymptote, as the slenderness ratio s is increased. It is noted that two pairs of lines cross, which shows that two mode shapes may exist at the same frequency. That is, the two mode shapes of 2nd and 3rd modes may exist where $c_2=c_3=41.43$ at $s=22.07$ (marked as ■). Also, the frequency curves of third and fourth modes come cross at (53.15, 103.4) (marked as ▲).

Figure 5 shows the effect of translational spring parameter $k_{\delta a}$ on the frequency ratio of c_i/c_{oi} when the clamped-clamped beam with $f=0.2$, $s=100$, $e=0.65$, $\mu=0.32$ and $s_p=71$ works loose its rigidity against immovability for the translational direction at the left end. Here, c_{oi} is the frequency parameter c_i of clamped-clamped beam and its corresponding values of c_{oi} are given in Fig. 5. The frequency ratio c_i/c_{oi} of imperfect clamped-clamped beam increases as $k_{\delta a}$ is

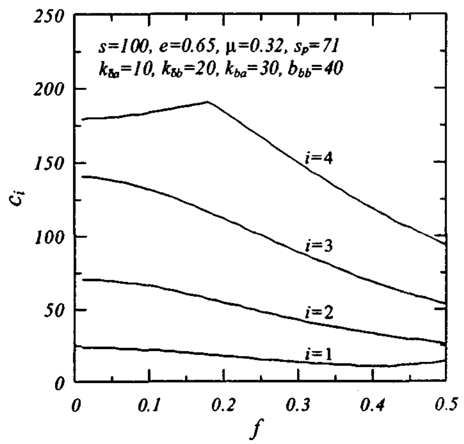

Fig. 3 c_i Versus f Curves

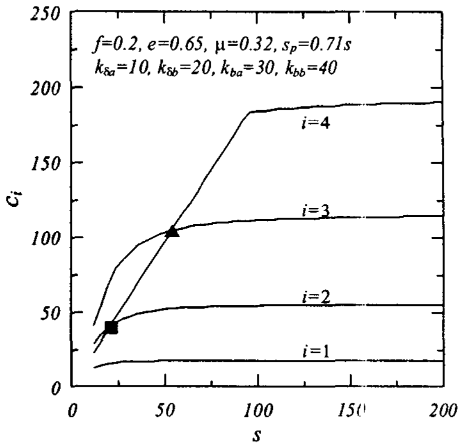

Fig. 4 c_i Versus s Curves

increased, as it is expected. As $k_{\delta a}$ gets smaller, especially when $k_{\delta a} \leq 40$, the ratio increases significantly. It should be noted that the effect of $k_{\delta a}$ is negligible in case of the first mode when $k_{\delta a} \leq 100$, since its increasing rate is very small as shown in the Fig. 5.

Figure 6 shows the effect of rotational spring parameter k_{ba} on the frequency ratio of c_i/c_{oi} when the clamped-clamped beam with $f=0.2$ and $s=100$ works loose it rigidity of anti-rotation at the left end. Also shown in the figure are the corresponding values of c_{oi} for perfect clamped-clamped beam. The frequency ratio c_i/c_{oi} of imperfect clamped-clamped beam increases, as k_{ba} is increased. As k_{ba} gets smaller the ratio increases relatively more significantly. However it should be noted that all the frequency ratios are the same nearly.

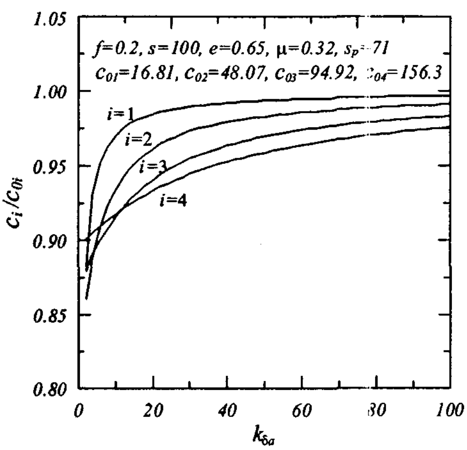

Fig. 5 Effects of $k_{\delta a}$ on c_i of Imperfect Clamped-Clamped Beam

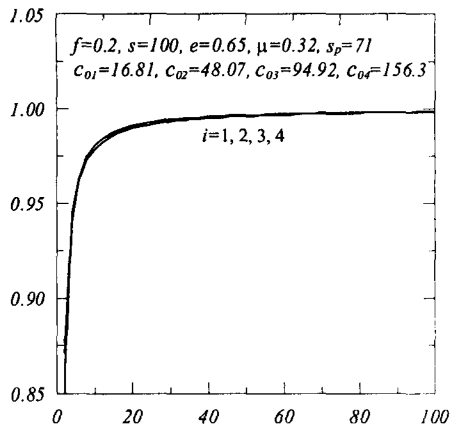

Fig. 6 Effects of k_{ta} on c_i of Imperfect Clamped-Clamped Beam

6. Concluding Remarks

This study deals with the free vibrations of horizontally curved beam with the general boundary condition. The boundary conditions of such beam are derived as the non-dimensional forms. Differential equations governing free vibrations of horizontally curved beams, adopted from the open literature[11], subjected to the boundary conditions derived herein were solved numerically to calculate both natural frequencies. For validating the theories and numerical methods presented herein, the frequency parameters are compared to those of SAP 2000. As the numerical results herein, the relationships between the frequency parameters and the various non-dimensional beam parameters are intensively investigated. It is expected that the results obtained herein can be practically utilized in the fields of vibration control.

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