자기접촉을 가지는 일래스티카 기둥의 수평좌굴 해의 간단한 해법

정재호, 강태진, 이경우*

서울대학교 공과대학 재료공학부, *동아대학교 패션 · 섬유학부

A Simple Solution of Lateral Buckling of an Elastica with Self-Contact

Jae Ho Jung, Tae Jin Kang, Kyung Woo Lee*

School of Materials Science and Technology, SNU, Korea

**Division of Fashion and Textile, Dong-A Univ. Busan, Korea*

Abstract

Exact Solution of lateral buckling of elastica column with self-contact was obtained. By assuming that there exists a counter force to maintain the horizontal coordinate at contact point to be the same value as that of highest point of bent beam, we could prevent the shape of solution from overlapping with each other. As the simpliest case, we obtained the elliptic integral solutions of the laterally post-buckled shape of linear elastic inextensible column under concentrated load. For this case, the slope angle at the contra-flexure point remains constant regardless of the value of load parameter beyond the minimum critical value. The counter force and other several parameters are expressed in terms of critical slope angle at the contra-flexure point. Finally it is suggested a numerical algorithms for solving the problem, which enable to solve the problem for nonlinear material or combined loading case.

1. Introduction

Lateral buckling of solid material frequently occurs in real life. Especially a fabric or film is likely to be on that situation. Post-buckled shape in one dimensional line does not need the concept of self-contact because it is assumed that the overlapping is freely occurred and slippage force at contact point is negligible. However, for relatively thick rods, fabrics or films, self-contact phenomenon should be considered. Although Grosberg and Swani[3], and Clapp and Peng[1, 2] dealt with the buckling problem of woven fabrics considering it as the linear elastic case, the self contact phenomenon was not taken in consideration. Joo et al[4] dealt with the same problem considering it as nonlinear elastic case, they also did not consider the self contact effect. Contact must occur at point region where the slope angle is perpendicular to that of initial axis of the film or fabric. It is possible that there is no shear effect at the contact point because the profile of buckled column will stay smooth after self-contact. Moreover, the two contacting bodies of the buckled column do not slip from each other as the two gears rotate in turn. Thus we can simplify the self-contact phenomenon of the lateral buckling of film or fabric. It is the aim of this paper to establish the simplified equation considering self-contact effect and to find the solution.

2. Mathematical Derivations

Figure 1 shows the shape of laterally buckled column neglecting self-contact effect.

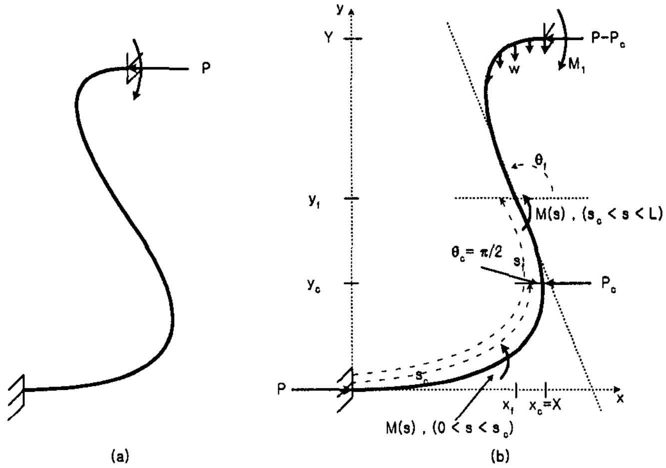

Fig 1. Simplified concept of self-contact for lateral buckling of a column.

Where, s_f , θ_f , x_f , y_f are the arc length, slope angle, x, and y coordinates at contra-flexure point. The normalized governing equation of this system is

$$\frac{d^2\theta}{d\xi^2} = -\beta\sin\theta + \gamma(1-\xi)\cos\theta \quad (0 \le \xi \le \xi_c) \tag{1a}$$

$$\frac{d^2\theta}{d\xi^2} = -(\beta-\beta_c)\sin\theta + \gamma(1-\xi)\cos\theta \quad (\xi_c \le \xi \le 1) \tag{1b}$$

$$\xi = \frac{s}{L},\ \bar{x} = \frac{x}{L},\ \bar{y} = \frac{y}{L},\ \overline{X} = \frac{X}{L},\ \overline{Y} = \frac{Y}{L},\ \beta = \frac{PL^2}{EI},\ \beta_c = \frac{P_cL^2}{EI},\ \gamma = \frac{wL^3}{EI}$$

The derived mathematical solutions have the form of elliptic integrals as follow.

$$\beta_c = \beta - \frac{\beta(A+2B)^2}{(\sqrt{2\beta}-A)^2} \quad A = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_f}},\ B = \int_{\frac{\pi}{2}}^{\theta_f} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_f}} \tag{2a,b,c}$$

$$\xi_c = \int_0^{\xi_c} d\xi = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2\beta(\cos\theta - \cos\theta_f)}} = \frac{A}{\sqrt{2\beta}} \tag{3a}$$

$$\xi_f = \xi_c + \int_{\frac{\pi}{2}}^{\theta_f} \frac{d\theta}{\sqrt{2(\beta-\beta_c)(\cos\theta - \cos\theta_f)}} = \frac{A}{\sqrt{2\beta}} + \frac{B(\sqrt{2\beta}-A)}{\sqrt{2\beta}(A+2B)} \tag{3b}$$

$$A = \sqrt{2}F(k,\ \arcsin(\frac{1}{k\sqrt{2}})),\ B = \sqrt{2}\{F(k,\ \frac{\pi}{2}) - F(k,\ \arcsin(\frac{1}{k\sqrt{2}}))\}$$

$$\bar{x}=\int_0^{\theta}\frac{\cos\theta d\theta}{\sqrt{2\beta(\cos\theta-\cos\theta_f)}} \qquad (0\le\xi\le\xi_c)$$

$$=\frac{1}{\sqrt{\beta}}\{2E(k\ ,\ \phi)-F(k\ ,\ \phi)\} \quad (0\le\phi\le\arcsin(\frac{1}{k\sqrt{2}})) \tag{4a}$$

$$\bar{x}=\int_0^{\frac{\pi}{2}}\frac{\cos\theta d\theta}{\sqrt{2\beta(\cos\theta-\cos\theta_f)}}+\int_{\frac{\pi}{2}}^{\theta}\frac{\cos\theta d\theta}{\sqrt{2(\beta-\beta_c)(\cos\theta-\cos\theta_f)}}=\frac{2E\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}-F\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}}{\sqrt{\beta}}$$

$$+\frac{2[E\{k\ ,\ \phi\}-E\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}]-[F\{k\ ,\ \phi\}-F\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}]}{\sqrt{(\beta-\beta_c)}}$$

$$(\xi_c\le\xi\le\xi_f) \quad or \quad (\arcsin(\frac{1}{k\sqrt{2}})\le\phi\le\frac{\pi}{2}) \tag{4b}$$

$$\bar{x}=\int_0^{\frac{\pi}{2}}\frac{\cos\theta d\theta}{\sqrt{2\beta(\cos\theta-\cos\theta_f)}}+\int_{\frac{\pi}{2}}^{\theta_f}\frac{\cos\theta d\theta}{\sqrt{2(\beta-\beta_c)(\cos\theta-\cos\theta_f)}}+\int_{\theta}^{\theta_f}\frac{\cos\theta d\theta}{\sqrt{2(\beta-\beta_c)(\cos\theta-\cos\theta_f)}}$$

$$=\frac{1}{\sqrt{\beta}}[2E\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}-F\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}]+\frac{1}{\sqrt{(\beta-\beta_c)}}\S 2[E\{k\ ,\ \frac{\pi}{2}\}-E\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}]$$

$$-[F\{k\ ,\ \frac{\pi}{2}\}-F\{k\ ,\ \arcsin(\frac{1}{k\sqrt{2}})\}]?+\frac{1}{\sqrt{(\beta-\beta_c)}}\ 2[E\{k\ ,\ \frac{\pi}{2}\}-E\{k\ ,\ \phi\}]-[F\{k\ ,\ \frac{\pi}{2}\}-F\{k\ ,\ \phi\}]$$

$$(\xi_f\le\xi\le 1) \quad or \quad (0\le\phi\le\frac{\pi}{2}) \tag{4c}$$

$$\bar{y}=\int_0^{\theta}\frac{\sin\theta d\theta}{\sqrt{2\beta(\cos\theta-\cos\theta_f)}} \qquad (0\le\xi\le\xi_c)$$

$$=\frac{2k}{\sqrt{\beta}}(1-\cos\phi) \qquad (0\le\phi\le\arcsin(\frac{1}{k\sqrt{2}})) \tag{5a}$$

$$\bar{y}=\int_0^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\sqrt{2\beta(\cos\theta-\cos\theta_f)}}+\int_{\frac{\pi}{2}}^{\theta}\frac{\sin\theta d\theta}{\sqrt{2(\beta-\beta_c)(\cos\theta-\cos\theta_f)}} \qquad (\xi_c\le\xi\le\xi_f)$$

$$=\frac{2k}{\sqrt{\beta}}[1-\cos\{\arcsin(\frac{1}{k\sqrt{2}})\}]+\frac{2k}{\sqrt{\beta-\beta_c}}(\cos\{\arcsin(\frac{1}{k\sqrt{2}})\}-\cos\phi) \qquad (\arcsin(\frac{1}{k\sqrt{2}})\le\phi\le\frac{\pi}{2}) \tag{5b}$$

$$\bar{y}=\int_0^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\sqrt{2\beta(\cos\theta-\cos\theta_f)}}+\int_{\frac{\pi}{2}}^{\theta_f}\frac{\sin\theta d\theta}{\sqrt{2(\beta-\beta_c)(\cos\theta-\cos\theta_f)}}+\int_{\theta}^{\theta_f}\frac{\sin\theta d\theta}{\sqrt{2(\beta-\beta_c)(\cos\theta-\cos\theta_f)}} \quad (\xi_f\le\xi\le 1)$$

$$=\frac{2k}{\sqrt{\beta}}[1-\cos\{\arcsin(\frac{1}{k\sqrt{2}})\}]+\frac{2k}{\sqrt{\beta-\beta_c}}\cos\{\arcsin(\frac{1}{k\sqrt{2}})\}+\frac{2k}{\sqrt{\beta-\beta_c}}\cos\phi \qquad (0\le\phi\le\frac{\pi}{2}) \tag{5c}$$

Where,

$$\theta_f = 117.54^\circ \ , \ k = 0.855092 \tag{6a}$$

$$\beta_{\min} = \{F(k \ , \ \pi)\}^2 = 18.0382 \tag{6b}$$

3. Results and Discussions

Figures 2 and 3 show half shape of the post-buckled column for given value of $\beta = 2(A+B)^2$ and $\beta = 2(A+B)^2 + 30$ by elliptic integral and numerical method.

$$0 = \beta - \frac{\beta(A+2B)^2}{(\sqrt{2\beta} - A)^2} \ , \ \beta_{\min} = 2(A+B)^2 \tag{7}$$

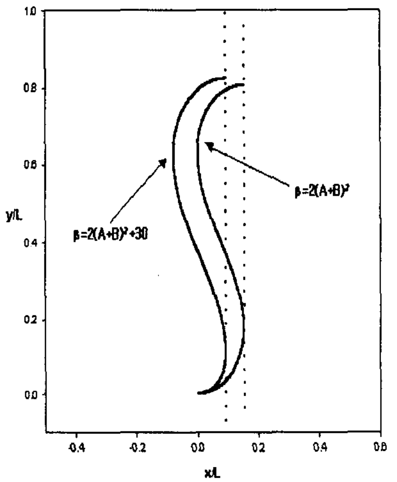

Figure 2. Half shape of the post-buckled column for given value by elliptic integral

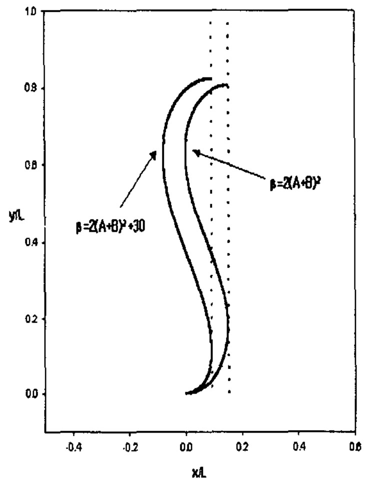

Figure 3. Half shape of the post-buckled column by Runge-Kutta method

4. Conclusions

Simplified governing equations and their solutions of laterally buckled elastica column considering self-contact were obtained. By assuming that there exists a counter force to maintain the horizontal coordinate at contact point to be the same value as that of highest point of bent beam, overlapping of equilibrium shape can be avoided. As the simplest case, we obtained the elliptic integral solutions of the laterally post-buckled shape of linear elastic inextensible column under concentrated load.

5. Literature Cited

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