비선형 우주척도인자 갖는 우주

Cosmology with non-smooth scale factor

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요약

이 논문에서는 우주의 진화와 관련하여 미분이 불가한 여러 시점을 포함한 경우의 곡률을 계산하기 위하여 비틀림적 공간에 적용하여 프리이드만 로버트슨 워커의 우주모델을 연구하였다. 특히 천체 물리학적인 관점에서본 복사-물질우세기, 물질-우주상수 우세기의 두점을 포함한 경우의 곡률을 계산하였다.

Abstract

In the framework of Lorentzian warped products, we study the Friedmann-Robertson-Walker cosmological model to investigate non-smooth curvatures associated with multiple discontinuities involved in the evolution of the universe. In particular we analyze non-smooth features of the spatially flat Friedmann-Robertson-Walker universe by introducing double discontinuities occurred at the radiation- matter and matter-lambda phase transitions in astrophysical phenomenology.

I. Introduction

Since the cosmic microwave background was discovered, there have been many ideas and proposals to figure out how the universe has evolved. The standard big bang cosmological model based on the Friedmann-Robertson-Walker(FRW) spacetimes has led to the inflationary cosmology [1] and nowadays to the M-theory cosmology with bouncing universes [2]. From a physical point of view, these warped product spacetimes are interesting since they include classical examples of spacetime such as the FRW manifold and the intermediate zone of Reissner-Nordström (RN) manifold [3, 4].

On the other hand, the concept of a warped product manifold was introduced by Bishop and O'Neill long ago [12], and it was later connected to general relativity [13] and semi-Riemannian geometry [14] by elevating warped products to a central role. Warped product spaces has been also extended to a richer class of spaces involving multiply product spaces [5, 6]. One of us has investigated the curvature of a multiply warped product possessing C^0-warping functions with a discontinuity at a single point [5], and in this paper we will generalize this result to a warped product spacetime with multiple discontinuities associated with cosmological phenomenology. Of particular interest are spacetimes with metric tensors which fail to be C^1 across multiple points on the hypersurface, and is C^∞ off the hypersurface. We will also study the Lorentzian metric which fails to be C^0 across multiple points on the hypersurfaces and is C^∞ off the hypersurfaces.

In this paper, as a cosmological model we will exploit the FRW spacetimes $M_0 \times_f H$, which can be treated as a warped product manifold possessing warping function (or scale factor) f with time dependence, to investigate the non-smooth curvature associated with the multiple discontinuities involved in the evolution of the universe. We will also analyze non-smooth features of the spatially flat FRW universe by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions in the astrophysical phenomenology.

In section We will study the realistic cosmological phenomenology in the spatially flat FRW universe associated with the radiation-matter and matter-lambda phase transitions in section 3.

II. FRW metric with multiple discontinuities

The FRW spacetime is one of the *warped product* manifold where the base is an open interval M_0 of R with usual metric reversed $(M_0, -dt^2)$, the fiber is a

3-dimensional Riemannian manifold (F, g_F) and the warping function f is any positive function f on M_0. The Robertson-Walker spacetime is then the product manifold $M = M_0 \times_f H$ endowed with the Lorentzian metric $g = -dt^2 + f^2(t)g_H$ with f being the scale factor of the FRW universe associated with universal expansion. This warping function f is a function of time alone and it measures how physical separations change with time. The dynamics of the expanding universe only appears implicitly in the time dependence of the warping function (or scalar factor) f.

Consider the spacetime $(M,\ g)$ with metric $g = -dt^2 + f^2 d\sigma^2$ in the form of warped products. Let $M = M_0 \times_f H$ be a warped product with $g_{M_0} = -dt^2$. Let $f > 0$ be smooth functions on $M_0 = (t_0,\ t_\infty)$. Assume $f \in C^\infty$ for $t \neq t_i$ and $f \in C^1$ at $t = t_i$ $(i = 1, 2, ..., n)$. When $f \in C^1$ at points $t_i \in (t_0,\ t_\infty)$ and $S = \{t_i\} \times_f H$, we define $f \in C^1(S)$ as a collection of functions $\{f^{(i)}\}$ with $f^{(i)}$ piecewisely defined on the intervals $t_i \leq t \leq t_{i+1}$ $(i = 0, 1, 2, ..., n)$ with $t_{n+1} = t_\infty$. Since $f \in C^1(S)$, we have $f^{(i-1)} = f^{(i)}$, $f^{(i-1)\prime} = f^{(i)\prime}$ but $f^{(i-1)\prime\prime} \neq f^{(i)\prime\prime}$. We shall use the unit step function μ for discontinuity of $f^{(i)\prime\prime}$ at $t = t_i$.

Consider the FRW metric of the form

$$g = -dt^2 + f^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

where k is a parameter denoting the spatially flat $(k = 0)$, 3-sphere $(k = 1)$ and hyperboloid $(k = -1)$ universes.

Proposition 2.1 Let $M = M_0 \times_f H$ be the FRW spacetime with Riemannian curvature R and flow vector field $U = \partial_t$. If $f \in C^1(S)$, vector fields $X, Y, Z \in \mathfrak{L}(H)$ satisfy

$$\begin{aligned}
(i) \quad & R_{XY}Z = \frac{f'^2 + k}{f^2}(\langle X,\ Z\rangle Y - \langle Y,\ Z\rangle X) \\
(ii) \quad & R_{XU}U = \frac{f''}{f}X \\
(iii) \quad & R_{XY}U = 0 \\
(iv) \quad & R_{XU}Y = \frac{f''}{f}\langle X,\ Y\rangle U
\end{aligned}$$

where f'' is given by

$$\begin{aligned}
f'' \;=\; & \left(f^{(n)\prime\prime} - f^{(n-1)\prime\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\right)\mu(t - t_n) \\
& + \sum_{l=1}^{n-1}\left(-f^{(l-1)\prime\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\right)\mu(t - t_l) \\
& + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\mu(t_n - t) \\
& + \sum_{l=1}^{n-1}\left(-f^{(l)\prime\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\right)\mu(t_l - t), \qquad (1)
\end{aligned}$$

with $\mu(t - t_i)$ being the unit step function which becomes unity for $t > t_i$ and vanishes otherwise.

Proof.

We derive f'' in terms of the collection of functions $\{f^{(i)}\}$ with $f^{(i)}$ piecewisely defined on the intervals $t_i \leq t \leq t_{i+1}$ $(i = 0, 1, 2, ..., n)$ with $t_{n+1} = t_\infty$. For a single discontinuity $n = 1$ case, f is trivially given by

$$f'' = f^{(1)\prime\prime}\mu(t - t_1) + f^{(0)\prime\prime}\mu(t_1 - t)$$

which fulfills (1). For double discontinuities $n = 2$ case, f is similarly given by

$$\begin{aligned}
f'' \;=\; & \left(f^{(2)\prime\prime} - \frac{1}{2}f^{(1)\prime\prime} + \frac{1}{2}f^{(0)\prime\prime}\right)\mu(t - t_2) \\
& + \left(\frac{1}{2}f^{(1)\prime\prime} - \frac{1}{2}f^{(0)\prime\prime}\right)\mu(t - t_1) \\
& + \left(\frac{1}{2}f^{(1)\prime\prime} + \frac{1}{2}f^{(0)\prime\prime}\right)\mu(t_2 - t) \\
& + \left(-\frac{1}{2}f^{(1)\prime\prime} + \frac{1}{2}f^{(0)\prime\prime}\right)\mu(t_1 - t), \qquad (2)
\end{aligned}$$

which also fulfills (1). By using iteration method, one can obtain (1) for an arbitrary n case. □

For the case of $f \in C^0(S)$ we use the derivative of the unit step function $\mu(t_i)$. For all $t \neq t_i$ this is well-defined, $\mu'(t) = 0$. However, at $t = t_i$ there exists a jump discontinuity so that we cannot define classical derivative and thus we use the δ-function, $\mu'(t - t_i) = \delta(t - t_i)$ to obtain the follow results.

Proposition 2.2 Let $M = M_0 \times_f H$ be the FRW spacetime with Riemannian curvature R and flow vector field $U = \partial_t$. If $f \in C^0(S)$, vector fields $X, Y,$

$Z \in \mathfrak{L}(H)$ then satisfy

$$(i) \quad R_{XY}Z = \frac{f'^2 + k}{f^2}(\langle X, Z\rangle Y - \langle Y, Z\rangle X)$$

$$(ii) \quad R_{XU}U = \frac{f''}{f}X$$

$$(iii) \quad R_{XY}U = 0$$

$$(iv) \quad R_{XU}Y = \frac{f''}{f}\langle X, Y\rangle U$$

where f' and f'' are given by

$$\begin{aligned} f' &= \left(f^{(n)\prime} - f^{(n-1)\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime}\right)\mu(t - t_n) \\ &+ \sum_{l=1}^{n-1}\left(-f^{(l-1)\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime}\right)\mu(t - t_l) \\ &+ \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime}\mu(t_n - t) \\ &+ \sum_{l=1}^{n-1}\left(-f^{(l)\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime}\right)\mu(t_l - t) \end{aligned} \tag{3}$$

$$\begin{aligned} f'' &= \left(f^{(n)\prime\prime} - f^{(n-1)\prime\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\right)\mu(t - t_n) \\ &+ \sum_{l=1}^{n-1}\left(-f^{(l-1)\prime\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\right)\mu(t - t_l) \\ &+ \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\mu(t_n - t) \\ &+ \sum_{l=1}^{n-1}\left(-f^{(l)\prime\prime} + \frac{1}{n}\sum_{k=0}^{n-1} f^{(k)\prime\prime}\right)\mu(t_l - t) \\ &+ \left(f^{(n)\prime} - f^{(n-1)\prime}\right)\delta(t - t_n) \\ &+ \sum_{l=1}^{n-1}\left(f^{(l)\prime} - f^{(l-1)\prime}\right)\delta(t - t_l), \end{aligned} \tag{4}$$

with $\mu(t-t_i)$ and $\delta(t-t_i)$ being the unit step function and the delta function, respectively.

Proof.
Similar to (1) in Proposition 3.1, one can readily obtain f'. Differentiating f' with respect to t and using the definition of the delta function $\mu'(t - t_i) = \delta(t - t_i)$ at $t = t_i$, one can also obtain f''. □

Proposition 2.3 Let $M = M_0 \times_f H$ be the FRW spacetime with Riemannian curvature R and flow vector field $U = \partial_t$. If $f \in C^0(S)$ and $X, Y \in \mathfrak{L}(H)$, then Ricci curvature is given by

$$(i) \quad \mathrm{Ric}(U,U) = -\frac{3f''}{f}$$

$$(ii) \quad \mathrm{Ric}(U,X) = 0$$

where f'' is given by (4).

Proposition 2.4 Let $M = M_0 \times_f H$ be the FRW spacetime with Riemannian curvature R and flow vector field $U = \partial_t$. If $f \in C^0(S)$, the Einstein scalar curvature is given by

$$R = 6\left(\frac{f'^2}{f^2} + \frac{f''}{f} + \frac{k}{f^2}\right),$$

where f' and f'' are given by (3) and (4).

Proposition 2.5 For every plane containing a vector field of $U = \partial_t$, if $f \in C^0(S)$ and $X, Y \in \mathfrak{L}(H)$, we have a sectional curvature K on the spacetime (M, g) for an arbitrary plane containing a vector field of $U = \partial_t$ and $W = \alpha U + \beta Y$

$$K(W,X) = \frac{-\alpha^2 f'' + \beta^2(f' + k)}{(-\alpha^2 + \beta^2)f^2}$$

where f' and f'' are given by (3) and (4).

Proof.
This follows $K(W,X) = \frac{g(R_{WX}W,X)}{g(W,W)g(X,X)-[g(W,X)]^2}$ of the nondegenerate 2-plane with basis (W, X). □

III. Cosmology of spatially flat FRW metric with double discontinuities

In the spatially flat FRW cosmology with $k = 0$, the early universe was radiation dominated, the adolescent universe was matter dominated, and the present universe is now entering into lambda-dominate phase in the absence of vacuum energy. If the universe underwent inflation, there was a very early period when the stress-energy was dominated by vacuum energy. The Friedmann equation may be integrated to give the age of the universe in terms of present cosmological parameters. We have the scale factor f as a function of time t which scales as $f(t) \propto t^{1/2}$ for a radiation-dominated (RD) universe, and scales as $f(t) \propto t^{2/3}$ for a matter-dominated (MD) universe, and scales as $f(t) \propto e^{Kt}$ for a lambda-dominated (LD) universe. Note that the transition from the radiation-dominated phase to the matter-dominated is not an abrupt one; neither is the later transition from the matter-dominated phase to the exponentially growing lambda-dominated phase.

With the above astrophysical phenomenology in mind, consider the spatially flat FRW spacetime (M, g) with metric $g = -dt^2 + f^2(t)d\sigma^2$ in the form of warped products. Let $M = M_0 \times_f H$ be a warped product with $g_{M_0} = -dt^2$.

Definition 3.1 A C^0-Lorentzian metric on M is a nondegenerate (0,2) tensor of Lorentzian signature such that

(i) $g \in C^0$ on S

(ii) $g \in C^\infty$ on $M \cap S^c$

(iii) for all $p \in S$, and $U(p)$ partitioned by S, $g|_{U_p^+}$ and $g|_{U_p^-}$ have smooth extensions to U. We call S a C^0-singular hypersurface of (M, g).

Consider M_0 as a C^0-singular hypersurface of (M, g). In the spatially flat FRW spacetime, $f > 0$ is smooth functions on $M_0 = (t_0, t_\infty)$ except at $t \neq t_i$ $(i = 1, 2)$, that is $f \in C^\infty(S)$ (where $S = \{t_i\} \times_f H$) for $t \neq t_i$ and $f \in C^0(S)$ at $t = t_i \in M_0$ to yield

$$(5) \qquad f = \begin{pmatrix} f^{(0)} = c_0 t^{1/2}, \text{ for } t < t_1 \\ f^{(1)} = c_1 t^{2/3}, \text{ for } t_1 \le t \le t_2 \\ f^{(2)} = c_2 e^{Kt}, \text{ for } t > t_2 \end{pmatrix}$$

with the boundary conditions

$$(6) \qquad c_0 t_1^{1/2} = c_1 t_1^{2/3}, \; c_1 t_2^{2/3} = c_2 e^{Kt_2}.$$

Experimental values for t_1 and t_2 are given by $t_1 = 4.7 \times 10^4$ yr and $t_2 = 9.8$ Gyr [15]. Moreover c_1 and c_2 are given in terms of c_0, t_1 and t_2 as follows

$$c_1 = c_0 t_1^{-1/6}, \; c_2 = c_0 t_1^{-1/6} t_2^{2/3} e^{-Kt_2}.$$

Note that in the spatially flat FRW model, $f \in C^0(S)$ since if we assume $f \in C^1(S)$ one could have the boundary conditions $\frac{1}{2}c_0 t_1^{-1/2} = \frac{2}{3}c_1 t_1^{-1/3}$ and $\frac{2}{3}c_1 t_2^{-1/3} = Kc_2 e^{Kt_2}$, which cannot satisfy the above boundary conditions (6) simultaneously.

Proposition 3.2 Let $M = M_0 \times H$ be the spatially flat FRW spacetime with Riemannian curvature R, flow vector field $U = \partial_t$ and warping function $f \in C^0(S)$. *For vector fields X, Y, $Z \in \mathfrak{L}(H)$ we have*

$$(i) \qquad R_{XY}Z = \frac{f'^2}{f^2}(\langle X, Z\rangle Y - \langle Y, Z\rangle X)$$

$$(ii) \qquad R_{XU}U = \frac{f''}{f}X$$

$$(iii) \qquad R_{XY}U = 0$$

$$(iv) \qquad R_{XU}Y = \frac{f''}{f}\langle X, Y\rangle U$$

where f is given by (5) and f' and f'' are given by

$$\begin{aligned} f' &= \left(\frac{1}{4}c_0 t^{-1/2} - \frac{1}{3}c_1 t^{-1/3} + Kc_2 e^{Kt}\right)\mu(t - t_2) \\ &+ \left(-\frac{1}{4}c_0 t^{-1/2} + \frac{1}{3}c_1 t^{-1/3}\right)\mu(t - t_1) \\ &+ \left(\frac{1}{4}c_0 t^{-1/2} + \frac{1}{3}c_1 t^{-1/3}\right)\mu(t_2 - t) \\ (7) \quad &+ \left(\frac{1}{4}c_0 t^{-1/2} - \frac{1}{3}c_1 t^{-1/3}\right)\mu(t_1 - t) \end{aligned}$$

$$\begin{aligned} f'' &= \left(-\frac{1}{8}c_0 t^{-3/2} + \frac{1}{9}c_1 t^{-4/3} + K^2 c_2 e^{Kt}\right)\mu(t - t_2 \\ &+ \left(\frac{1}{8}c_0 t^{-3/2} - \frac{1}{9}c_1 t^{-4/3}\right)\mu(t - t_1) \\ &+ \left(-\frac{1}{8}c_0 t^{-3/2} - \frac{1}{9}c_1 t^{-4/3}\right)\mu(t_2 - t) \\ &+ \left(-\frac{1}{8}c_0 t^{-3/2} + \frac{1}{9}c_1 t^{-4/3}\right)\mu(t_1 - t) \\ &+ \left(-\frac{2}{3}c_1 t^{-1/3} + Kc_2 e^{Kt}\right)\delta(t - t_2) \\ (8) \quad &+ \left(-\frac{1}{2}c_0 t^{-1/2} + \frac{2}{3}c_1 t^{-1/3}\right)\delta(t - t_1), \end{aligned}$$

with $\mu(t - t_i)$ and $\delta(t - t_i)$ being the unit step function and the delta function, respectively.

Proof.

Substituting f in (5) into (3) and (4) in Proposition 3.2, one can readily obtain (7) and (8). □

Proposition 3.3 Let $M = M_0 \times H$ be the spatially flat FRW spacetime with Riemannian curvature R, flow vector field $U = \partial_t$ and warping function $f \in C^0(S)$. For vector fields X, Y, $Z \in \mathfrak{L}(H)$, the Ricci curvature is given by

$$(i) \qquad \mathrm{Ric}(U, U) = -\frac{3f''}{f}$$

$$(ii) \qquad \mathrm{Ric}(U, X) = 0$$

$$(iii) \qquad \mathrm{Ric}(X, Y) = \left(\frac{2f'^2}{f^2} + \frac{f''}{f}\right)\langle X, Y\rangle,$$

if $X, Y \perp U$

where f, f' and f'' are given by (5), (7) and (8), respectively.

Proposition 3.4 Let $M = M_0 \times H$ be the spatially flat FRW spacetime with Riemannian curvature R, flow vector field $U = \partial_t$ and warping function $f \in C^0(S)$. The Einstein scalar curvature is then given by

$$R = 6\left(\frac{f'^2}{f^2} + \frac{f''}{f}\right),$$

where f, f' and f'' are given by (5), (7) and (8), respectively.

Proposition 3.5 For every plane containing a vector field of $U = \partial_t$ and $f \in C^0(S)$, if $X, Y \in \mathfrak{L}(H)$ we have a sectional curvature K on the FRW spacetime (M, g) for an arbitrary plane containing a vector field of $U = \partial_t$ and $W = \alpha U + \beta Y$

$$K(W, X) = \frac{-\alpha^2 f'' + \beta^2 f'}{(-\alpha^2 + \beta^2) f^2}$$

where f, f' and f'' are given by (5), (7) and (8), respectively.

Proposition 3.6 Let $M = M_0 \times H$ be the spatially flat FRW spacetime with Riemannian curvature R, flow vector field $U = \partial_t$ and warping function $f \in C^0(S)$. The evolution equations are then given by

$$(i)\ \frac{3f'^2}{f^2} = 8\pi\rho + \Lambda$$

$$(ii)\ \frac{3f''}{f} = -4\pi(\rho + 3P) + \Lambda,$$

where f, f' and f'' are given by (5), (7) and (8), respectively. Here ρ, P and Λ are the mass density and pressure of matter and the cosmological constant.

Proof.
Consider the Einstein equation

$$(9)\quad G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor, and $T_{\mu\nu}$ is the stress-energy tensor for all the field present-matter, radiation and so on. To be consistent with the symmetries of the metric, the total stress-energy tensor $T_{\mu\nu}$ must be diagonal, and by isotropy the spatial components must be equal. The simplest realization of such a stress-energy tensor is that of a perfect fluid characterized by a time-dependent energy density $\rho(t)$ and pressure $p(t)$,

$$(10)\qquad T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p).$$

Substituting (10) into (9), together with the Ricci and Einstein curvatures given in Proposition 4.3 and Proposition 4.4, one can readily obtain the above evolution equations. □

Remarks 3.7 The $\mu = 0$ component of the conservation of stress-energy tensor, $T^{\mu\nu}_{;\nu} = 0$, gives the first law of thermodynamics of the familiar form $d(\rho f^3) = -pd(f^3)$ or equivalently, $d[f^3(\rho + p)] = f^3 dp$. The change in energy in a co-moving volume element, $d(\rho f^3)$, is equal to minus the pressure times the change in volume element, $-pd(f^3)$. For the simple equation of state $p = \omega\rho$, where ω is independent of time, the energy density evolves as $\rho \propto f^{-3(1+\omega)}$. Examples of interest include: radiation ($p = \frac{1}{3}\rho$, $\rho \propto f^{-4}$), matter ($p = 0$, $\rho \propto f^{-3}$) and vacuum energy ($p = -\rho$, $\rho \propto$ const.) phases.

IV. Conclusions

We have considered the FRW cosmological model in the warped product scheme to investigate the non-smooth curvature associated with the multiple discontinuities involved in the evolution of the universe. In particular we have analyzed the non-smooth features of the spatially flat FRW universe by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions in the astrophysical phenomenology.

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