다원환의 비가환 미분가군

Noncommutative Derivation Modules of Algebras

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요약

본 논문은 미분가군이 가환대수에서 정의된 미분가군과 임의의 다원환에 대하여 정의된 Bergman 의 관점에서 본 미분가군으로서 보편적 미분가군과 다른 구조를 갖는 미분가군 동형사상에 의하여 유일하게 결정될 수 있다는 것을 밝히는 것이다.

Abstract

We construct a universal derivation module of an R-algebra which is constructed by means of a tensor product of three copies of the algebras.

I. Introduction

There are two different notions of derivation modules of algebras. One notion of derivation modules is defined only for commutative algebras, which is called a commutative derivation modules. The other notion of derivation modules is defined for algebras(not necessarily commutative), which is called a non-commutative derivation module. In this paper, several properties of universal derivation modules (non-commutative derivation modules) in the sense of [1] is investigated.

Let R be a commutative ring with unity 1, and A a unitary R-algebra that is not necessarily commutative. A derivation module of A is a pair (M, d), where M is an (A, A)-bimodule and $d: A \rightarrow M$ is an R-derivation, i.e. an R-linear mapping such that $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$. For two derivation modules (M, d) and (N, δ), an (A, A)-bimodule homomorphism forms a category.

The natural question arises that is there are universal object in the category of an algebra. As an partial answer of this question, in this theses we study the following. We construct a universal derivation module of an R-algebra which is constructed by means of a tensor product of three copies of the algebras.

II. Preliminaries

Derivation modules investigated in this thesis are different from derivation modules defined only for commutative algebras. But the technique used in this thesis and the nature of problems dealt with are rather similar to those for commutative algebras in [4] and [6] then for non-commutative case in[1]. Therefore, an understanding of well known results for commutative case will be stated.

Let R be a commutative ring with unity 1, and A a commutative algebra over R. An R-linear mapping $d: A \rightarrow M$, where M is an A-module , is called an R-derivation of A if $d(ab) = ad(b) + d(a)b$, for all $a, b \in A$. An R-derivation $d: A \rightarrow A$ is called an R - derivation on A. A pair (M, d) is called a derivation module of A if M is an A-module and $d: A \to M$ is an R-derivation. For any two derivation modules (M, d) and (N, δ), an A-module homomorphism $f: M \to N$ such that $f \circ d = \delta$ is called a derivation module homomorphism. A derivation module (U, d) is called a universal derivation module of A if for any derivation module (M, δ) of A, there exists a unique derivation module homomorphism $f: (U, d) \to (M, \delta)$.

It is well known ([4],[5],[6],[13]) that for any commutative R-algebra A, there exists a universal derivation module of A, and it is unique up to unique derivation module isomorphism. In fact, let $U = A \otimes_R A / J$, where J is the A-submodule of $A \otimes_R A$ generated by all elements of the form

$1 \otimes ab - a \otimes b - b \otimes a$, $a, b \in A$ and define $d: A \to U$ by $d(a) = \nu(1 \otimes a)$, $a \in A$, where ν is the natural homomorphism from $A \otimes_R A$ into U, then (U, d) is a universal derivation module of A. On the other hand, let $I = \ker \pi$ and $V = A \otimes_R A / I^2$, where π: $A \otimes_R A \to A$ is an R-algebra homomorphism defined by $a \otimes b \mapsto ab$ for all $a, b \in A$, Then (V, δ), where $\delta: A \to V$ is an R-derivation defined by $\delta(a) = 1 \otimes a - a \otimes 1 + I^2$, for $a \in A$, is a universal derivation module of A. We will list some basic properties of universal derivation module of a commutative algebras.

Proposition 2.1 ([5]). Let $R[X]$ be a polynomial ring over R with X as a set of indeterminates. If U is a free $R[X]$-module with a set $\{u_x \mid x \in X\}$, where $u_x = u_y$ iff $x = y$, for $x, y \in X$, as a basis, and $d: R[X] \to U$ is an R-derivation defined by $df = \sum_{x \in X} (\partial f / \partial x) u_x$ for $f \in R[X]$, then (U, d) is a universal derivation module of $R[X]$.

Proposition 2.2 ([4]). Let A be a free commutative join of a family $(A_\alpha)_{\alpha \in I}$ of its subalgebras, and (U_α, d_α) a universal derivation module of A_α for each $\alpha \in I$. Let

(1) $U = \oplus_{\alpha \in I} (A \otimes_{A_\alpha} U_\alpha)$, and

(2) $d: A \to U$ is an R-derivation defined by

$$d(\sum_\alpha a_{\alpha_1} \cdots a_{\alpha_n}) = (\sum_{j=1} a_{\alpha_1} \cdots \hat{a}_{\alpha_j} \cdots a_{\alpha_n} \otimes d_{\alpha_j}(a_{\alpha_j})),$$

where $a_{\alpha_i} \in A_{\alpha_i}$, $\alpha_i \in I$, and $\hat{a}_{\alpha_j}$ denotes the omission of a_{α_j}. Then (U, d) is a universal derivation module of A.

Let M be a direct sum of R-modules M_α of its submodules, and A and B the R-algebras which contain M. Let the A_α and B_α be the subalgebras of A and B, respectively, generated by M_α for each $\alpha \in I$.

Proposition 2.3 ([6]). Let A be a free join of $(A_\alpha)_{\alpha \in I}$ of its subalgebras, and let $h: A \to B$ be a algebra homomorphism such that $h \mid M$ is the identity map on M. If $h_\alpha: A_\alpha \to B$ be a mapping such that $h_\alpha = h \mid A_\alpha$ for each $\alpha \in I$, then B is a free join of a family $(B_\alpha)_{\alpha \in I}$ iff h is onto and $\ker h$ is the ideal of A generated by $\sum_{\alpha \in I} \ker h_\alpha$.

III. Non Commutative Derivation Modules

Construction 3.1. Let A be an R-algebras with unity 1, A tensor product $A \otimes_R A \otimes_R A$ of A can be considered as an (A, A)-bimodule with the left scalar multiplication given by

$$a(\sum a_i \otimes b_i \otimes c_i) = \sum aa_i \otimes b_i \otimes c_i, \quad (1)$$

and the right scalar multiplication given by

$$(\sum a_i \otimes b_i \otimes c_i) b = \sum a_i \otimes b_i \otimes c_i b \quad (2)$$

for all $a, b \in A$ and $\sum a_i \otimes b_i \otimes c_i \in A \otimes_R A \otimes_R A$. These are well defined.

Let ϕ_a be the R-linearization of ϕ_a' for all $a \in I$. If we define the left multiplication by $ax = \phi_a(x)$ for all $a \in A$ and $x \in A \otimes_R A \otimes_R A$, then the scalar multiplication is well defined, and same as in (1). Similarly, the right scalar multiplication same as in (2) is well defined. Let J be an (A, A)-submodule of the (A, A)-bimodule $A \otimes_R A \otimes_R A$ generated by all elements of the type

$$1 \otimes ab \otimes 1 - a \otimes b \otimes 1 - 1 \otimes a \otimes b$$

for all $a, b \in A$. Let $U = (A \otimes_R A \otimes_R A)/J$, and $d: A \to U$ an R-linear mapping defined by $d(a) = 1 \otimes a \otimes 1 + J$, for all $a, b \in A$,

$$d(ab) = ad(b) + d(a)b .$$

Hence d is an R-derivation. To show that (U, d) is a universal derivation module of A, let (M, δ) be derivation module of A. Let $\phi' : A \times A \times A \to M$ be an R-multilinear map defined by $(a, b, c) \mapsto a\delta(b)c$ for all $a, b, c \in A$, and let ϕ_1 be the R-linearization of ϕ'. Then ϕ_1 is an (A, A)-bimodule homomorphism such that $J \subseteq \ker \phi_1$.

Let $\phi: U \to M$ be a derivation module homomorphism of derivation modules of A. Since every element of U can be denoted by

$$\sum a_i \otimes b_i \otimes c_i + J = \sum a_i d(b_i) c_i .$$

U is an (A, A)-bimodule generated by $d(A) = \{d(a) \mid a \in A\}$. If $\Psi: U \to M$ is any derivation module homomorphism, then it is clear that $\phi = \Psi$, since

$$\Psi(\sum a_i \otimes b_i \otimes c_i + J) = \sum a_i \delta(b_i) c_i .$$

Then (U, d) constructed in this way is a universal derivation module of A.

Construction 3.2. Let A be an R-algebras with unity 1. Consider an R-linear mapping $\pi: A \otimes_R A \to A$ given by $a \otimes b \mapsto ab$ for all $a, b \in A$. This map is well defined. In fact, let $\pi' : A \times A \to A$ be an R-bilinear map given by $(a, b) \mapsto ab$ for all $a, b \in A$. Let $\pi: A \otimes_R A \to A$ be the R-linearization of π'. Then π is an (A, A)-bimodule homomorphism, since $\pi(a(\sum a_i \otimes b_i) b) = a(\pi(\sum a_i \otimes b_i))b$, for all $a, b \in A$ and $\sum a_i \otimes b_i \in A \otimes_R A$. Let $U = \ker \pi$, and define a mapping $d: A \to U$ by $d(a) = 1 \otimes a - a \otimes 1$, $a \in A$. Then d is an R-derivation, since

$$d(ra + sb) = rd(a) + sd(b) ,$$

$d(ab) = d(a)b + ad(b)$, for all $a, b \in A$. Let (M, δ) be any derivation module of A, and $\phi': A \times A \to M$ is an R-bilinear mapping given by $(a, b) \mapsto a\delta(b)$, $a \in A$.

Let ϕ_1 be the R-linearization of ϕ', and let $\phi = \phi_1 \mid U$. It is easy to show that ϕ is an (A, A)-bimodule such that $\phi \circ d = \delta$. For any $\sum a_i \otimes b_i \in U$, $\sum a_i \otimes b_i = \sum a_i d(b_i)$.

This implies that U is an (A, A)-bimodule generated by $d(A)$. To show that (U, d) is a universal derivation module of A, consider a derivation module homomorphism $\Psi: (U, d) \mapsto (M, \delta)$. Then ϕ must be Ψ, since U is generated by $d(A)$. We proved that thus (U, d) constructed in this way is a universal derivation module of A.

Example 3.3. Let F be a field, and

$$A = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

be the matrix algebra over F. If $U = \{\sum A_i \otimes A_j \in A \otimes_R A \mid \sum A_i A_j = 0\}$ and $\delta: A \to U$ is a F-derivation given by

$\delta(M) = I \otimes M - M \otimes I,$

where $M \in A$ and I is the identity matrix in A. In fact ,

$$\delta\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

for some $x, y, z \in F \otimes_F F$ such that $x + y + z = d(a + b + c)$, where d is the universal derivation of F given by Construction 3.2.

Remark 3.4. From Construction 3.2., we know that every elements of any universal derivation module (U, d) of an R-algebra A can be expressed of the form $\sum a_i d(b_i) = -\sum d(a_i) b_i$,

where $\sum a_i b_i = 0$ for all $a_i, b_i \in A$.

Theorem 3.5. Let $f: A \to B \to 0$ be an exat sequence of algebra over R, and let (U, d) and (V, δ) be universal derivation modules of A and B , respectively. If

$$J = U(\ker f) + (\ker f)U + Ad(\ker f) + d(\ker f)A,$$ and

$\partial : B \to U/J$ is an R-derivation defined by $\partial(b) = d(a) + J$ for some $a \in f^{-1}(b)$, then $(U/J, \partial)$ is isomorphic to (V, δ) as a derivation module of B.

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