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Decisions under risk and uncertainty through the use of Choquet integral

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Abstract

The Choquet-Stieltjes integral is defined. It is shown that the Choquet -Stieltjes integral is represented by a Choquet integral. As an application of the theorem above, it is shown that Choquet expected utility model for decision under uncertainty and rank dependent utility model for decision under risk are respectively same as their simplified version.

Keywords : Fuzzy measure, Non-additive measure, Choquet integral, Decision under uncertainty. Decision under risk

1 Introduction

About the decision theory under uncertainty, the expected utility theory by von Neumann and Morgenstern [5] is well known. However, in recent years the counterexample that human's decisions do not follow the expected utility theory is reported in various literatures. The Choquet integral with respect to non additive set function, which is called with various names, (e.g. fuzzy measure, non-additive measure, capacity, non-additive subjective probability,) is a basic tool for modeling of decisions under risk and uncertainty. We can explain the famous paradoxes, that is, Allais paradox [1] and Ellsberg's paradox [6] by using Choquet integral model. To explain Allais paradox for decision under risk, Rank dependent utility model are proposed by Quiggin [9]. For Ellsberg's paradox, which is relevant to decision under uncertainty, Choquet expected utility model is proposed by Schmeidler [11]. After that, the simplified version in which the utility function is not used, is proposed by Chateauneuf [2].

In this paper we define the Choquet-Stieltjes integral and show that the Choquet-Stieltjes integral is represented by a Choquet integral. As an application of theorem above, we show that Choquet expected utility model (resp. rank dependent utility model) is same as its simplified version.

The structure of the paper is as follows. In section 2, we define the fuzzy measure and Choquet integral and show their basic properties. We present the Choquet integral representation theorem of comonotonically additive functional. In Section 3, we introduce two famous counterexamples of classical expected utility models and define some non-expected utility models. In Section 4 we define Choquet-Stieltjes integral and show that the Choquet-Stieltjes integral is represented by a Choquet integral with respect to another fuzzy measure. As the corollary of the theorem, we show that Choquet expected utility model (resp. rank dependent utility model) is same as its simplified version. We show that the simplified version is sufficient to explain the paradoxes mentioned above.

2 Fuzzy measure and Choquet integral

In this section, we define fuzzy measure, the Choquet integral and show their basic properties.

Let S be a universal set and $\mathcal{S}$ be $\sigma-$ algebra of S, that is, $(S,\mathcal{S})$ be a measurable space.

Definition 2.1. [13] Let $(S,\mathcal{S})$ be a measurable space. A fuzzy measure μ is a real valued set function, $\mu : \mathcal{S} \longrightarrow R^+$ with the following properties;

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in \mathcal{S}$.

We say that the triplet $(S, \mathcal{S}, \mu)$ is a fuzzy measure space if μ is a fuzzy measure.

$\mathcal{F}(S)$ denotes the class of non-negative measurable functions, that is,

$$\mathcal{F}(S) = \{f | f : S \to R^+, f : \text{measurable}\}$$

Definition 2.2. [3, 7] Let $(S, \mathcal{S}, \mu)$ be a fuzzy measure space. The Choquet integral of $f \in \mathcal{F}(S)$ with respect to μ is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{x | f(x) \geq r\})$.

Suppose that $S = \{1, 2, \ldots, n\}$. The i-th order statistic $x^{(i)}$ [14] is a functional on $[0,1]^n$ which is defined by arranging the components of $x = (x_1, \cdots, x_n) \in [0,1]^n$ in the increasing order

$$x^{(1)} \leq \cdots \leq x^{(i)} \leq \cdots \leq x^{(n)}.$$

Using the i-th order statistics, the Choquet integral is written as

$$(C) \int x d\mu = \sum_{i=1}^{n} (x^{(i)} - x^{(i-1)}) \mu(\{(i) \cdots (n)\}),$$

where we define $x^{(0)} := 0$.
Definition 2.3. [4] Let $f, g \in \mathcal{F}(S)$. We say that f and g are comonotonic if

$$f(x) < f(x') \Rightarrow g(x) \leq g(x')$$

for $x, x' \in S$.
Definition 2.4. Let I be a real-valued functional on $\mathcal{F}(S)$. We say I is comonotonically additive if and only if $I(f+g) = I(f) + I(g)$ for comonotonic $f, g \in \mathcal{F}(S)$, and I is monotone if and only if $f \leq g \Rightarrow I(f) \leq I(g)$ for $f, g \in \mathcal{F}(S)$.

Next we present that the comonotonically additive functional I on $\mathcal{F}$ which satisfies a less restrictive condition than monotonicity can be represented by the Choquet integral.

Definition 2.5. [8] We say that a functional on $\mathcal{F}(S)$ is comonotonic monotone if $f \leq g$ implies $I(f) \leq I(g)$ for comonotonic $f, g \in \mathcal{F}(S)$.

In the following we suppose that the functional I on $\mathcal{F}(S)$ is comonotonically additive and comonotonic monotone (for short c.a.c.m.). The next theorem is less restrictive version of Schmeidler's representation theorem [11].
Theorem 2.6. I is a c.a.c.m. functional on $\mathcal{F}$ if and only if there exists a fuzzy measure μ such that

$$I(f) = (C) \int f d\mu$$

for all $f \in \mathcal{F}(S)$.

3 Decision under risk and uncertainty

In this section we present frames for decision under risk and uncertainty and paradoxes that the classical expected utility theory fails. Next we present the definitions of the Choquet expected utility and the Rank dependent utility.

Let S be a state space and X be a set of outcomes. We assume that outcomes are monetary. Therefore we may suppose $X \subset R$. We mean, by "decision under uncertainty", situations when there does not exist a given objective probability. In decision under uncertainty, we consider the set of function f from S to X: we say the function f the act. $\mathcal{F}$ denotes the set of acts, that is,

$$\mathcal{F} = \{f | f : S \longrightarrow X\}.$$

$\prec$ denotes the weak order on $\mathcal{F}$. We say that the quadruplet $(S, X, \mathcal{F}, \prec)$ is the frame for decision under uncertainty.

In contrast with decision under uncertainty, by "decision under risk, we mean situations when there exists a objective probability on S. In decision under risk, we consider the set $\mathcal{P}$ of probability on S and the set of function f from S to X: we say the function f the random variable. $\mathcal{F}$ denotes the set of random variable. $\prec$ denotes the weak order on $\mathcal{F}$. We say that the quintuplet $(S, X, \mathcal{P}, \mathcal{F}, \prec)$ is the frame for decision under uncertainty.

The next example is a famous paradox that fails the expected utility theory of von Neumann and Morgenstern [5]
Example 1. (Allais paradox [1]) Let the state space $S := \{1, 2, \ldots, 100\}$ and the outcome $X := \{0, 100, 200\}$. Define random variables f_1,f_2,f_3 and f_4 by $f_1(x) := 100$ for all
$x \in S$, $f_2(x) := \begin{cases} 200 & 1 \leq x \leq 70 \\ 0 & 71 \leq x \leq 100, \end{cases}$,
$f_3(x) := \begin{cases} 100 & 1 \leq x \leq 15 \\ 0 & 16 \leq x \leq 100 \end{cases}$
and $f_4(x) := \begin{cases} 200 & 1 \leq x \leq 10 \\ 0 & 11 \leq x \leq 100. \end{cases}$ f_1 means that you always get 100 dollars. f_2 means that you will get 200 dollars if you take a number 1 to 70 and you get none if you take a number 71 to 100. It is reported that not a few people choose f_1, that is, $f_2 \prec f_1$. In the same way f_3 means that you will get 100 dollars if you take number 1 to 15, and f_4 means that you will get 200 dollars if you take the number 1 to 10. It is reported that not a few people chose f_4, that is, $f_3 \prec f_4$ If the probabilities are all the same, i.e. $P(i) = \frac{1}{100}$ for all $i \in S$, this preference cannot be repre-

sented by expected utility. In fact, suppose that there exists a utility function $u : X \to R^+$ such that $f \prec g \Leftrightarrow E(u(f)) < E(u(g))$, where $E(\cdot)$ is the classical expectation. Since $f_2 \prec f_1$, we have $0.7u(200) < u(100)$. On the other hand, it follows from $f_3 \prec f_4$ that $0.15u(100) \prec 0.1u(200)$. Therefore we have $1.05u(100) < 0.7u(200) < u(100)$. This is a contradiction.

The next example is Ellesberg's paradox [6] for decision under uncertainty.

Example 2. (Ellesberg's paradox)
Consider the red and black and white ball in the urn. The number of red is 30, black and white is 60. The number of Black is unknown. f_R means that you will get 100 dollars only if you take red ball and f_B means that you will get 100 dollars only if you take black ball. Not a few people select f_R because there may be a few black ball in the urn that is, $f_B \prec f_R$. f_{RW} means that you will get 100 dollars if you take red or white ball and f_{BW} means that you will get 100 dollars if you take black or white ball. Not a few people select f_{BW} because there may be a few white ball in the urn, that is, $f_{RW} \prec f_{BW}$. This preference cannot be explained by the expected utility theory. In fact, let the state space $S := \{R, B, W\}$ and the set of outcome $X := \{0, 100\}$. The acts f_R, f_B, f_{RW} and f_{BW} are defined by the table below. Suppose

	30	60	
	Red	Black	White
f_R	\$ 100	0	0
f_B	0	\$ 100	0
f_{RW}	\$ 100	0	\$ 100
f_{BW}	0	\$ 100	\$ 100

that there exists a subjective probability P such that $f \prec g \Leftrightarrow E(u(f)) < E(u(g))$. It follows from $f_{RW} \prec f_{BW}$ that $u(100)P(B) = u(100)(P(BW) - P(W)) > u(100)(P(RW) - P(W)) = u(100)P(R)$. This contradicts $f_B \prec f_R$.

To solve those paradoxes, the model using Choquet integral with respect to fuzzy measure has been proposed; that is, the Choquet expected utility model for decision under uncertainty and the rank dependent utility model for decision under risk. First we define the Choquet expected utility model (CEU) for decision under uncertainty, that is introduced by Schmeidler [12].

Definition 3.1. Consider the frame of decision under uncertainty. The Choquet expected utility model stipulates that the decision maker ranks act f with the help of a utility function $u : R \to R$, which is continuous and strictly increasing. The ranking $C_{u,\mu}$ of acts f is performed through

$$C_{u,\mu}(f) := (C) \int u(f) d\mu,$$

that is, $f \prec g \Leftrightarrow I(f) < I(g)$, where μ is a fuzzy measure (capacity).

Next we define the Rank dependent expected utility model (RDEU) by Quiggin [9].
Definition 3.2. Consider the frame of decision under risk. A decision maker behaves in accordance with the rank dependent expected utility model if the decision maker's preferences $\prec$ are characterized by two functions u and w: a continuous and strictly increasing function $u : R \to R$ and a probability distorting function w such that $f \succ g \Leftrightarrow J(f) > J(g)$ where $J(h) := (C) \int u(h) d(w \circ P)$ for a random variable h.

The next section we will present the solution of the paradoxes using simplified CEU and RDEU model.

4 Choquet-Stieltjes integral

In this section we define the Choquet-Stieltjes integral and show that Choquet-Stieltjes integral is represented by Choquet integral. Applying the theorem above, we show that the CEU and the RDEU are same as their simplified version by Chateauneuf [2].
Definition 4.1. Let $(S, \mathcal{S}, \mu)$ be a fuzzy measure space and $\varphi : R^+ \to R^+$ be a non-decreasing real valued function. Then we can define Lebesgue-Stjeltjes measure ν_{φ_f} [10] on real line by

$$\nu_{\varphi_f}([a,b]) := \varphi_f(b+0) - \varphi_f(a-0)$$

$$\nu_{\varphi_f}((a,b)) := \varphi_f(b-0) - \varphi_f(a+0).$$

We define Choquet-Stieltles integral $CS_{\mu,\varphi}(f)$ with respect to μ by

$$CS_{\mu,\varphi}(f) := \int_0^\infty \mu_f(r) d\nu_\varphi(r),$$

where $\mu_f(r) = \mu(\{x | f(x) \geq r\})$.

If the space $S = \{1, 2, \ldots, n\}$, Using the i-th order statistics, the Choquet-Stieltjes integral is written as

$$CS_{\mu,\varphi}(f) = \sum_{i=1}^{n} (\varphi(x^{(i)}) - \varphi(x^{(i-1)}) \mu(\{(i) \cdots (n)\}).$$

Since Choquet-Stieltjes integral is comonotonically addtive and comonotonically monotone, applying the representation theorem (Theorem 2.6), we have the next theorem.

Theorem 4.2. Let $(S, \mathcal{S}, \mu)$ be a fuzzy measure space and $\varphi : R^+ \to R^+$ be a non decreasing function. There exists a fuzzy measure $\nu_{\mu,\varphi}$ such that

$$CS_{\mu,\varphi}(f) = (C) \int f d\nu_{\mu,\varphi},$$

that is, the Choquet-Stieltjes integral can be represented by Choquet integral.

Suppose that φ is strictly increasing, Since $\{x | f(x) > \varphi^{-1}(\alpha)\} = \{x | \varphi(f(x)) > \alpha\}$, we have

$$(C) \int \varphi(f) d\mu = CS_{\mu,\varphi}(f).$$

Therefore we have the next corollary.

Corollary 4.3. Let $(S, \mathcal{S}, \mu)$ be a fuzzy measure space and $\varphi : R^+ \to R^+$ be a strictly increasing function. Then there exists a fuzzy measure $\nu_{\mu,\varphi}$ such that

$$(C) \int \varphi(f) d\mu = (C) \int f d\nu_{\mu,\varphi}.$$

The corollary above means that the CEU (resp. the RDEU) are same as its simplified version. Using the simplified version, we can solve both Allais' and Ellsberg's paradoxes.

Example 3.

(i) (Allais paradox) We can define the probability distortion function $w : [0,1] \to [0,1]$ such that $w(0.1) = 0.08$, $w(0.15) = 0.1$ $w(0.7) = 0.45$ and $w(1) = 1$. Then it follows from the Choquet integral C_μ with respect to the fuzzy measure $\mu := w \circ P$ that $C_\mu(f_1) = 100 > C_\mu(f_2) = 90$ and $C_\mu(f_3) = 10 < C_\mu(f_4) = 16$.

(ii) (Ellsberg's paradox) We may define the fuzzy measure such that $\mu(\{R\}) := 1/3$, $\mu(\{B\}) = \mu(\{W\}) := 2/9$,
$\mu(\{R, W\}) := 5/9$,
$\mu(\{B, W\}) = \mu(\{R, B\}) := 2/3$ and $\mu(\{R, B, W\}) = 1$. Then we have the Choquet integral C_μ of f_* by the table below. The

f_*	f_R	f_B	f_{RW}	f_{BW}
$C_\mu(f_*)$	1/3	2/9	5/9	2/3

above table says that $C_\mu(f_B) < C_\mu(f_R)$ and $C_\mu(f_{RW}) < C_\mu(f_{BW})$. Then there is no contradiction.

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