2003 International Symposium on Advanced Intelligent Systems
September 25-28, 2003, Jeju, Korea

A Concept of Fuzzy Wavelets based on Rank Operators and Alpha-Bands

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***Abstract*—** A concept of fuzzy wavelets is proposed by a fuzzification of morphological wavelets. In the proposed fuzzy wavelets, analysis and synthesis schemes can be formulated as the operations of fuzzy relational calculus. In order to perform an efficient compression and reconstruction, an alpha-band is also proposed as a soft thresholding of the wavelets. In the image compression/reconstruction experiment using test images extracted Standard Image DataBAse (SIDBA), it is confirmed that the root mean square error (RMSE) of the proposed soft thresholding is decreased to 87.3% of the conventional hard thresholding.

1 Introduction

A fuzzification of wavelets, i.e., fuzzy wavelets have been presented in [2] [3]. In [2] [3], an image compression and reconstruction based on the fuzzy wavelets is formulated as two dimensional signal interpolation by Takagi-Sugeno (T-S) fuzzy system. Whereas, Heijmans et al [1] have presented a wavelet decomposition scheme by means of the mathematical morphology and rank operators, i.e., morphological wavelets. From a broad perspective, the former is based on the space-frequency structure and the latter is based on the ordered structure.

This paper proposes a concept of fuzzy wavelets based on mathematical morphology and rank operators, that is, it corresponds to a fuzzification of the morphological wavelets [1]. By this fuzzification, many schemes of the fuzzy sets theory, specially fuzzy relational calculus can be applied to the wavelets. In order to perform an efficient image compression and reconstruction, an alpha-band (an extension of alpha-cut) is also proposed as a soft threshold scheme in the fuzzy wavelets.

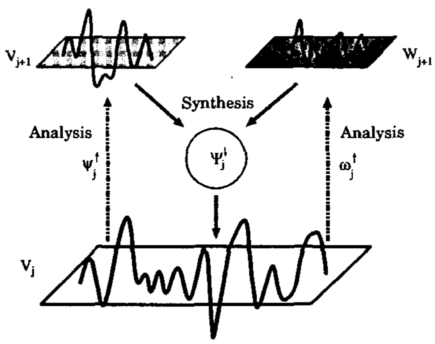

Figure 1: Overview of decomposition scheme

2 Morphological Wavelets

Let V_j and W_j be the signal space at level j and the detail space at level j, respectively. Signal analysis consists of decomposing a signal in direction of increasing j by means of signal analysis operators $\psi_j^\uparrow : V_j \to V_{j+1}$, and detail analysis operators $\varpi_j^\uparrow : V_j \to W_j$. We have addition and subtraction operators $\dot{+}$, $\dot{-}$ on V_j such that $x_1 \dot{+} (x_2 \dot{-} x_1) = x_2$, for $x_1, x_2 \in V_j$. Given an input signal $x_0 \in V_0$, we consider the following recursive signal analysis scheme, called the pyramid transform:

$$\begin{aligned} x_0 &\to \{x_1, y_0\} \to \{x_2, y_1, y_0\} \to \cdots \\ &\to \{x_{k+1}, y_k, y_{k-1}, \cdots, y_0\}, \to \cdots \end{aligned} \quad (1)$$

where

$$x_{j+1} = \psi_j^\uparrow(x_j) \in V_{j+1}, \quad (2)$$

and

$$y_j = \varpi_j^\uparrow(x_j) \in W_j, \quad (3)$$

The original signal $x_0 \in V_0$ can be exactly reconstructed from x_{k+1} and $y_0, y_1, \ldots, y_k$ by means of the backward recursion $x_j = \psi_j^\downarrow(x_{j+1}) \dot{+} y_j, j = k, k-1, \cdots, 0$. The two dimensional case is as follows. By $\mathbf{n}$, $2\mathbf{n}$ we denote the points $(m,n), (2m,2n) \in \mathbf{Z}_x \times \mathbf{Z}_y$, $\mathbf{Z}_x = \{1, 2, \ldots, Z_x\}$, $\mathbf{Z}_y = \{1, 2, \ldots, Z_y\}$, respectively, and by $2\mathbf{n}_+$, $2\mathbf{n}^+$, $2\mathbf{n}_+^+$ the points $(2m+1, 2n)$, $(2m, 2n+1)$,

$(2m+1, 2n+1)$, respectively. The analysis operators are defined as

$$\begin{aligned}\psi^{\uparrow}(\mathbf{n}) &= x(2\mathbf{n}) \wedge x(2\mathbf{n}_{+}) \wedge x(2\mathbf{n}^{+}) \wedge x(2\mathbf{n}_{+}^{+}),\\ \varpi^{\uparrow}(\mathbf{n}) &= (\varpi_v(\mathbf{n}), \varpi_h(\mathbf{n}), \varpi_d(\mathbf{n})),\end{aligned} \tag{4}$$

where the operator $\wedge$ denotes 'min' and ϖ_v, ϖ_h, ϖ_d represent the vertical, horizontal, and diagonal detail signals, given by

$$\begin{aligned}\varpi_v(x)(\mathbf{n}) &= \frac{1}{2}\left(x(2\mathbf{n}) - x(2\mathbf{n}^{+}) + x(2\mathbf{n}_{+}) - x(2\mathbf{n}_{+}^{+})\right),\\ \varpi_h(x)(\mathbf{n}) &= \frac{1}{2}\left(x(2\mathbf{n}) - x(2\mathbf{n}_{+}) + x(2\mathbf{n}^{+}) - x(2\mathbf{n}_{+}^{+})\right),\\ \varpi_d(x)(\mathbf{n}) &= \frac{1}{2}\left(x(2\mathbf{n}) - x(2\mathbf{n}_{+}) - x(2\mathbf{n}^{+}) - x(2\mathbf{n}_{+}^{+})\right).\end{aligned} \tag{5}$$

The synthesis operators are given by

$$\begin{aligned}\psi^{\downarrow}(2\mathbf{n}) &= \psi^{\downarrow}(x)(2\mathbf{n}_{+}) = \psi^{\downarrow}(x)(2\mathbf{n}^{+})\\ &= \psi^{\downarrow}(x)(2\mathbf{n}_{+}^{+}) = x(\mathbf{n}),\end{aligned} \tag{6}$$

and

$$\begin{aligned}\varpi^{\downarrow}(y)(2\mathbf{n}) &= (y_v(\mathbf{n}) + y_h(\mathbf{n})) \vee (y_v(\mathbf{n}) + y_h(\mathbf{n}))\\ &\quad \vee (y_h(\mathbf{n}) + y_d(\mathbf{n})) \vee 0,\\ \varpi^{\downarrow}(y)(2\mathbf{n}_{+}) &= (y_v(\mathbf{n}) - y_h(\mathbf{n})) \vee (y_v(\mathbf{n}) - y_d(\mathbf{n}))\\ &\quad \vee (-y_h(\mathbf{n}) - y_d(\mathbf{n})) \vee 0,\\ \varpi^{\downarrow}(y)(2\mathbf{n}^{+}) &= (y_h(\mathbf{n}) - y_v(\mathbf{n})) \vee (y_v(\mathbf{n}) - y_d(\mathbf{n}))\\ &\quad \vee (y_h(\mathbf{n}) - y_d(\mathbf{n})) \vee 0,\\ \varpi^{\downarrow}(y)(2\mathbf{n}_{+}^{+}) &= -(y_v(\mathbf{n}) - y_h(\mathbf{n})) \vee (y_d(\mathbf{n}) - y_v(\mathbf{n}))\\ &\quad \vee (y_d(\mathbf{n}) - y_h(\mathbf{n})) \vee 0,\end{aligned} \tag{7}$$

where the operator $\wedge$ denotes 'max' and we write $y \in W_1$ as $y = (y_v, y_h, y_d)$.

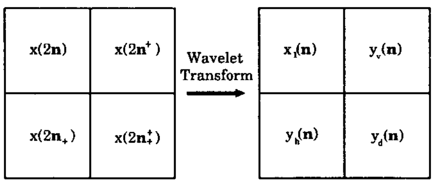

Figure 2: Two-dimensional wavelet transform an input signal x to a scaled signal x_1 and the vertical, horizontal, and diagonal signals, y_v, y_h, y_d, respectively.

Example)

By using the test images extracted from Standard Image DataBAse (SIDBA), 'Lenna' (Fig. 3, left), the example of decomposed images obtained by Eqs. (4) - (7), are shown. Figure 3 (right) corresponds to the signal and detailed image. The histogram of frequency with respect to intensity of signal space x_1. horizontal space y_h. vertical space y_v. diagonal space y_d. are shown in Figs. 4.

Figure 3: Original image 'Lenna' (left) and the decomposed image (right)

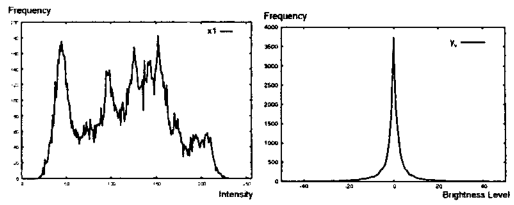

Figure 4: Frequency histogram with respect to intensity, x_1 (left), y_v (right), of 'Lenna'

In the case of the wavelets, an image compression and reconstruction can be achieved by thresholding the frequency histogram shown in Fig. 5. The thresholding of the histogram corresponds to replacing the pixel which is greater than the threshold value, by zero, it corresponds to an ordinal hard thresholding.

3 Proposed Fuzzy Wavelets

By using a normalization of the original image $x_0 \in V$, of the size $Z_x \times Z_y$, i.e., normalizing the range of the brightness level $\{0, \ldots, 255\}$ into a closed unit interval $[0, 1]$ $(= U)$, the original signal x_0 can be regarded as a fuzzy relation $x_0 \in F(\mathbf{Z}_x \times \mathbf{Z}_y)$, $\mathbf{Z}_x = \{1, \ldots, Z_x\}$, $\mathbf{Z}_y = \{1, \ldots, Z_y\}$. The transformed signals $\{x_{k+1}, y_k, y_{k-1}, \ldots, y_0\}$ also correspond to fuzzy relations by normalizing the brightness level into U. The analysis and synthesis schemes (defined as Eqs. (1) - (7)) can be used in the setting of the fuzzy relations. In this paper, an alpha-band is proposed as a soft threshold scheme of the wavelets, in order to perform an efficient image compression and reconstruction.

[Alpha-band (Hard-thresholding)]

The alpha-band is a sub-closed interval of U defined as

$$[a_l . a_u] \subset U, \tag{8}$$

where $a_l . a_u (\in U)$.

By using the alpha-band, the thresholding of the fuzzy relation x_0 is performed as follows:

$$\tilde{x_0}(\mathbf{n}) = \begin{cases} x_0(\mathbf{n}) & if \quad x_0(\mathbf{n}) \in [a_l, a_u], \\ 0 & otherwise. \end{cases} \quad (9)$$

The filtering process based on the alpha-band is equivalent to the ordinal hard-thresholding as follows:

$$Hist_{x_0}(u) = \begin{cases} Hist_{x_0}(u) & if \quad u \in [a_l, a_u], \\ 0 & otherwise, \end{cases} \quad (10)$$

where $Hist_{x_0}$ denotes the frequency histogram of the pixels of the fuzzy relation x_0 with respect to the brightness level u. This correspondence is shown in Figs. 5. The image compression and reconstruction can be achieved by using the thresholding shown in Eqs. (9) and (10).

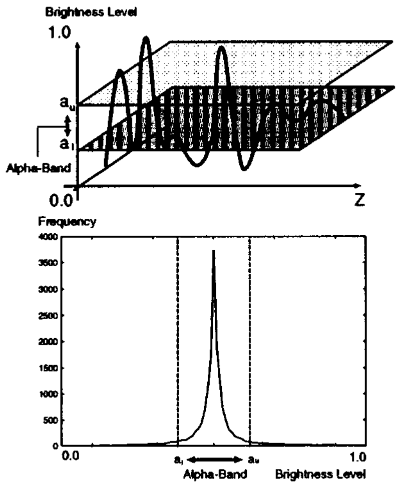

Figure 5: Correspondence of the filtering of fuzzy relation (upper) and histogram (lower)

The alpha-band scheme corresponds to the ordinal thresholding of the wavelets as previously stated in Sec. 2. In this paper, a soft thresholding of the wavelets is proposed by extending the parameters $a_l, a_u (\in U)$ to fuzzy numbers. The fuzzy numbers $\tilde{a}_l, \tilde{a}_u (\in F(U))$ are defined as three tuples

$$\tilde{a}_l = (\tilde{a}_l^{(l)}, \tilde{a}_l^{(c)}, \tilde{a}_l^{(u)}), \quad (11)$$

$$\tilde{a}_u = (\tilde{a}_u^{(l)}, \tilde{a}_u^{(c)}, \tilde{a}_u^{(u)}), \quad (12)$$

where $\tilde{a}_*^{(l)}$, $\tilde{a}_*^{(c)}$, and $\tilde{a}_*^{(u)}$ denote the lower bound, the center point, and the upper bound of the fuzzy number $\tilde{a}_*$, respectively, and the example is shown in Fig. 6.

[Soft Alpha-band (Soft-thresholding)]

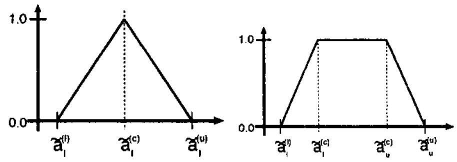

Figure 6: Example of definitions of fuzzy number of alpha-band (left) and interval (right)

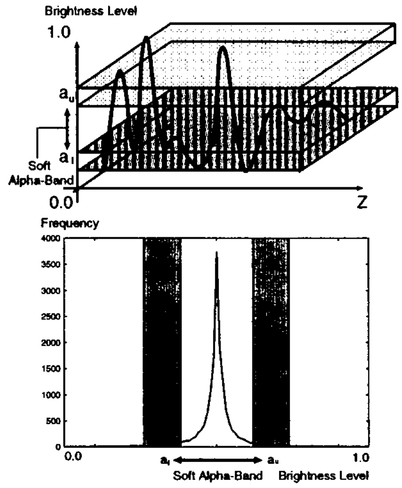

Figure 7: Correspondence of the soft filtering of fuzzy relation (upper) and histogram (lower)

The soft alpha-band is a fuzzy sub-interval of $F(U)$ defined as

$$[\tilde{a}_l, \tilde{a}_u] \subset F(U), \quad (13)$$

where the parameters $\tilde{a}_l, \tilde{a}_u \in F(U)$. An example of the soft alpha-band is shown in Fig. 6. By using the soft alpha-band, the filtering of the fuzzy relation x_0 is performed as follows:

$$\tilde{x}_0(\mathbf{n}) = \begin{cases} x_0(\mathbf{n}) & if \quad x_0(\mathbf{n}) \in [\tilde{a}_l^{(c)}, \tilde{a}_u^{(c)}], \\ Soft_l(x_0(\mathbf{n})) & if \quad x_0(\mathbf{n}) \in [\tilde{a}_l^{(l)}, \tilde{a}_l^{(c)}], \\ Soft_u(x_0(\mathbf{n})) & if \quad x_0(\mathbf{n}) \in [\tilde{a}_u^{(c)}, \tilde{a}_u^{(u)}], \\ 0 & otherwise, \end{cases} \quad (14)$$

where $Soft_l(x_0(\mathbf{n}))$ and $Soft_u(x_0(\mathbf{n}))$ produce $x_0(\mathbf{n})$ with being the probability $p \in [0, 1]$ defined as

$$p = \begin{cases} \frac{\tilde{a}_l^{(c)} - x_0(\mathbf{n})}{\tilde{a}_l^{(c)} - \tilde{a}^{(l)}{}_l} & if \quad Soft_l(x_0(\mathbf{n})), \\ \frac{\tilde{a}_u^{(u)} - x_0(\mathbf{n})}{\tilde{a}_u^{(u)} - \tilde{a}^{(c)}{}_u} & if \quad Soft_u(x_0(\mathbf{n})). \end{cases} \quad (15)$$

The proposed thresholding can improve the quality of the reconstructed image compared with the hard thresholding. The other types of fuzzy numbers (Gaussian etc) are used as the bound of the alpha-interval. The fuzzy sets theory can be applied to the soft thresholding expressed by the fuzzy numbers, and many improvements also may be proposed in terms of the human perspectives and subjective.

4 Experimental Comparison of Soft-thresholding and Hard-thresholding

In the experiment of image compression and reconstruction, a comparison of the hard thresholding and the soft thresholding is performed. The conditions of the soft thresholding are defined as Condition 1 : $\tilde{a}_l = (\tilde{a}_l^{(c)} - 0.04, \tilde{a}_l^{(c)}, \tilde{a}_l^{(c)} + 0.04)$, $\tilde{a}_u = (\tilde{a}_u^{(c)} - 0.04, \tilde{a}_u^{(c)}, \tilde{a}_u^{(c)} + 0.04)$, Condition 2 : $\tilde{a}_l = (\tilde{a}_l^{(c)} - 0.08, \tilde{a}_l^{(c)}, \tilde{a}_l^{(c)} + 0.08)$, $\tilde{a}_u = (\tilde{a}_u^{(c)} - 0.08, \tilde{a}_u^{(c)}, \tilde{a}_u^{(c)} + 0.08)$, and Condition 3 : $\tilde{a}_l = (\tilde{a}_l^{(c)} - 0.12, \tilde{a}_l^{(c)}, \tilde{a}_l^{(c)} + 0.12)$, $\tilde{a}_u = (\tilde{a}_u^{(c)} - 0.12, \tilde{a}_u^{(c)}, \tilde{a}_u^{(c)} + 0.12)$, respectively. The test image of the size 256×256 pixels is shown in Fig. 3. Fig 8 shows the result of measurement of Root Mean Square Errors (RMSE) with respect to the file size of compressed image, where the file size is adjusted by changing the values $\tilde{a}_l^{(c)}$ and $\tilde{a}_u^{(c)}$. As can be seen form Fig 8, the

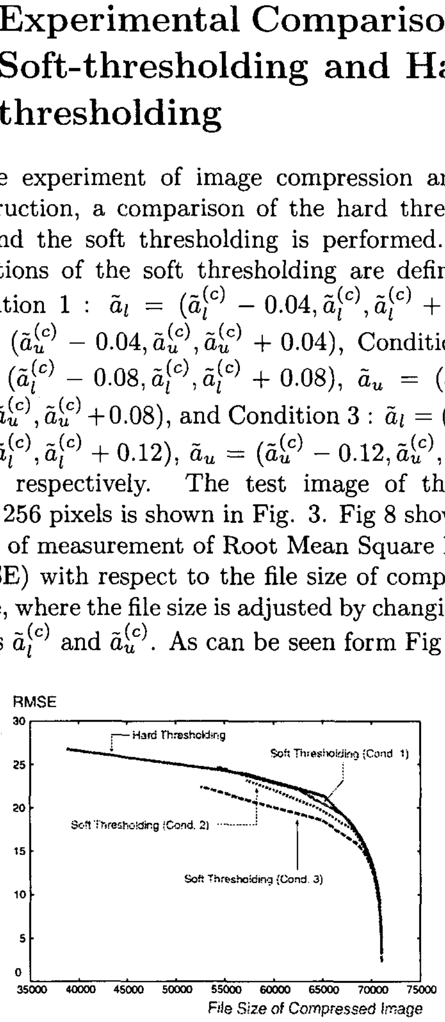

Figure 8: RMSE comparison with respect to the file size of compressed image

effectiveness of the soft thresholding (specially, the condition-3) is confirmed. Figure 9 shows the reconstructed images obtained by the hard thresholding and the soft thresholding with the compression size being the same. It is confirmed the effectiveness of the proposed thresholding as shown in Fig. 9.

5 Conclusions

A concept of fuzzy wavelets has been proposed as a fuzzification of morphological wavelets. In the proposed fuzzy wavelets, analysis and synthesis schemes

Figure 9: Reconstructed images, hard threshold (left: RMSE = 21.23), soft threshold with condition 3 (right: RMSE = 18.53)

can be formulated as the operations of fuzzy relational calculus, i.e., the proposed fuzzy wavelets is based on the ordered structure. In order to perform an efficient compression and reconstruction, an alpha-band is also proposed as a soft thresholding of the wavelets. In an experiment using test images extracted Standard Image DataBAse (SIDBA), the effectiveness of the proposed soft thresholding has been confirmed. Specially, under the condition that the file size of the compressed image is the same, the Root Mean Square Errors of the reconstructed image obtained by the proposed soft thresholding is decreased to 87.3 % of the conventional hard thresholding.

In the proposed soft-threshold scheme, the other types of fuzzy numbers (Gaussian etc) are used as the bound of the alpha-interval. Furthermore, the fuzzy sets theory can be applied to the soft thresholding expressed by the fuzzy numbers, and many improvements also may be proposed in terms of the human perspectives and subjective.

Acknowledgements

Support from is 'The Telecommunications Advancement Foundation' gratefully acknowledged.

References

[1] H. J. A. M. Heijmans, and J. Goutsias, "Nonlinear Multiresolution Signal Decomposition Schemes - Part II: Morphological Wavelets", *IEEE Transaction on Image Processing*, Vol. 9, No. 11, pp. 1897 - 1913, 2000.

[2] M. Thuillard, "Wavelet in Soft Computing", *World Scientific*.

[3] Y. Yu and S. Tan, "Stable Construction of Multi-Scale Fuzzy Wavelet System for Image Recovery and Compression", Advanced Signal Processing Algorithms, Architectures, and Implementations VI, (Ed. F. T. Luk), SPIE, Bellingham, 1998.