Step Length를 이용한 비비례감쇠시스템의 고유치 해석
Application of Step Length Technique To An Eigensolution Method for Non-proportionally Damped Systems

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ABSTRACT

This paper presents an efficient eigensolution method for non-proportionally damped systems. The proposed method is obtained by applying the accelerated Newton-Raphson technique and the orthonormal condition of the eigenvectors to the linearized form of the quadratic eigenproblem. A step length and a selective scheme are introduced to increase the convergence of the solution. The step length can be evaluated by minimizing the norm of the residual vector using the least square method. While the singularity may occur during factorizing process in other iteration methods such as the inverse iteration method and the subspace iteration method if the shift value is close to an exact eigenvalue, the proposed method guarantees the nonsingularity by introducing the orthonormal condition of the eigenvectors, which can be proved analytically. A numerical example is presented to demonstrate the effectiveness of the proposed method.

Keywords: *non-proportionally damped system, eigenvalue problem, step length, Newton-Raphson technique*

1. INTRODUCTION

The eigenvalue problem of the system should be solved *a priori* to avoid a resonance or to define the dynamic characteristics such as natural frequencies and mode shapes if the mode superposition method is used in the dynamic analysis of structures. In most analyses recently employed, the proportional damping that satisfies a condition developed by Caughey and O'Kelly (1965) is assumed for lack of more realistic representation. That is, the damping of

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the structure is assumed to be such that the free modes of vibration of the damped system are identical to those for the undamped structure. Under this assumption, one can straightly solve the eigenproblem in a low cost. However, in most real systems, the damping is non-proportional. Even when proportional damping is assumed for each sub-system in the analysis of soil-structure systems, composite structures, etc., the resulting damping for the complete structure will be non-proportional.

The common approach to solve the quadratic eigenvalue problem is to reformulate the quadratic equations into a linear one by doubling the order of the system. Many eigensolution methods have been proposed. Transformation methods such as QR (Moler and Stewart 1973), LZ (Kaufman 1974) or Jacobi (Veselic 1983) determine all the eigenpairs in an arbitrary sequence. This is not efficient when only few low frequencies are required in a large system. Moreover, since the initial matrices are modified during the solution process, these methods cannot fully take advantage of the sparseness of the matrices.

The Perturbation method (Meirovitch and Ryland 1979; Cronin 1990; Kwak 1993; Peres-Da-Silva et al. 1995; Tang and Wang 1995) sets the eigensolution of the undamped system as zero-order approximation and lets the higher-order terms account for the slight damping effect. It is very practical for slight damping case, since weakly damping implies that the eigensolution will differ only a little from that of the corresponding undamped system.

Gupta (1974, 1981), Utku and Clemente (1984) proposed a procedure combining the Sturm sequence and inverse iteration scheme to solve the linearized eigenproplem of spinning structures. The procedure preserves the banded nature of the matrices and is well suited for finding those frequencies, which fall within a certain range of interest. Despite the fact that the method is useful to solve a small number of desired modes, it requires many complex operations for each eigenvalue.

The subspace iteration method (Bathe and Wilson 1972; Chen and Taylor 1986; Leung 1995) combines the inverse iteration method, simultaneous iteration method and Rayleigh-Ritz analysis. It is a more efficient alternative algorithm than the inverse iteration procedure. The method employs nth order submatrices of the augmented linear eigenproblem in the iteration process by taking the block-partitioned nature of the matrices of the linearized problem. All required modes are solved simultaneously thus the round-off errors can be minimized. However, it requires a great deal of complex arithmetic operations.

On the other hand, Lanczos method was first proposed for undamped systems (Lanczos 1950; Paige 1971, 1972, 1976; Parlett and Scott 1979; Simon 1984), and extended to damped systems (Parlett et al. 1985; Kim and Craig 1988; Rajakumar and Rogers 1991; Rajakumar 1993; Chen and Taylor 1988; Chen 1994). The two-sided-Lanczos algorithm (Parlett et al. 1985; Kim and Craig 1988; Rajakumar and Rogers 1991; Rajakumar 1993) requires the generation of two sets of Lanczos vectors, left and right, and the symmetric Lanczos algorithm (Chen and Taylor 1988; Chen 1994) uses a set of Lanczos vectors to reduce a large eigenvalue problem in a much smaller one. Although only real arithmetic is used in the

solution process, in contrast to the case of real symmetric eigenproblems, there will be a possibility of a serious breakdown and the accuracy of the solutions obtained is low (Zheng et al. 1997).

Recently, Lee et al. (1998) proposed an efficient solution method to improve the numerical stability and increase the convergence by applying the modified Newton-Raphson technique and the orthonormal condition of the eigenvectors. This study further improves the convergence of the method by adopting the step length which can be evaluated using the least squares technique. In the following section, the basic concept of the proposed method is presented. In section 3, the efficiency of the proposed method is shown by analyzing a numerical example.

2. METHOD OF ANALYSIS

2.1 Problem Definition

The equation for free vibration of a linear time-invariant system of order n is written as

$$\mathbf{M}\ddot{u} + \mathbf{C}\dot{u} + \mathbf{K}u = 0 \tag{1}$$

where **M**, **C** and **K** are (n x n) mass, damping and stiffness matrices of the system, respectively, and u is the (n x 1) vector of system displacements. The damping matrix satisfying

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \tag{2}$$

is said to be proportional (Caughey and O'Kelly 1965). However, in most real systems, the damping matrix does not satisfy Eq. (2), that is, it is non-proportional. The eigenanalysis for such systems is traditionally performed in the space extended to $2n$-dimension such as

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} \tag{3}$$

where λ and ϕ are eigenvalue and associated eigenvector of the system, respectively. Eq. (3) may be written as

$$\mathbf{A}\psi = \lambda\mathbf{B}\psi \tag{4}$$

with

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} \tag{5}$$

Since both matrices **A** and **B** are not positive definite although they are symmetric, in general, the eigenvalues and the associated eigenvectors are complex values.

2.2 Modified Newton-Raphson method (Lee et al. 1998)

Suppose that initial approximate solutions $\lambda^{(0)}$ and $\psi^{(0)}$ of the eigenvalue and the associated

eigenvector of Eq. (5) are known. Denote the approximate eigenvalue after k iterations by $\lambda^{(k)}$ and its associated eigenvector by $\boldsymbol{\psi}^{(k)}$, and define the residual vector as follows

$$\mathbf{r}^{(k)} = \mathbf{A}\boldsymbol{\psi}^{(k)} - \lambda^{(k)}\mathbf{B}\boldsymbol{\psi}^{(k)} \tag{6}$$

The approximate eigenvector $\psi^{(k)}$ is then orthonormalized with respect to matrix **B**, such as

$$\left(\boldsymbol{\psi}^{(k)}\right)^T \mathbf{B}\boldsymbol{\psi}^{(k)} = 1 \tag{7}$$

Let the increment of the approximate eigenvalue from step k to step (k+1) be $\Delta\lambda^{(k)}$, and the increment vector of the approximate eigenvector from step k to step (k+1) be $\Delta\boldsymbol{\psi}^{(k)}$. Then we have

$$\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)} \tag{8}$$

$$\boldsymbol{\psi}^{(k+1)} = \boldsymbol{\psi}^{(k)} + \Delta\boldsymbol{\psi}^{(k)} \tag{9}$$

After (k+1) iterations, the residual vector can be written as

$$\mathbf{r}^{(k+1)} = \mathbf{A}\boldsymbol{\psi}^{(k+1)} - \lambda^{(k+1)}\mathbf{B}\boldsymbol{\psi}^{(k+1)} \tag{10}$$

where $\boldsymbol{\psi}^{(k+1)}$ also satisfies the orthonormal condition with respect to matrix **B** as follows

$$\left(\boldsymbol{\psi}^{(k+1)}\right)^T \mathbf{B}\boldsymbol{\psi}^{(k+1)} = 1 \tag{11}$$

Substituting Eqs. (8), (9) into Eq. (10), we can have

$$\mathbf{r}^{(k+1)} = [\mathbf{A} - (\lambda^{(k)} + \Delta\lambda^{(k)})\mathbf{B}]\left(\boldsymbol{\psi}^{(k)} + \Delta\boldsymbol{\psi}^{(k)}\right) \tag{12}$$

To get the solution converged to the eigenvalue and its associated eigenvector, we expect the residual vector to be a null vector, such as

$$[\mathbf{A} - (\lambda^{(k)} + \Delta\lambda^{(k)})\mathbf{B}]\left(\boldsymbol{\psi}^{(k)} + \Delta\boldsymbol{\psi}^{(k)}\right) = 0 \tag{13}$$

Introducing Eqs. (6) to (9) and neglecting the high order terms, namely, $\Delta\lambda^{(k)}\mathbf{B}\Delta\psi^{(k)}$ and $\left(\Delta\psi^{(k)}\right)^T \mathbf{B}\Delta\psi^{(k)}$, Eqs. (11) and (13) can be rewritten as

$$\left(\mathbf{A} - \lambda^{(k)}\mathbf{B}\right)\Delta\boldsymbol{\psi}^{(k)} - \Delta\lambda^{(k)}\mathbf{B}\boldsymbol{\psi}^{(k)} = -\mathbf{r}^{(k)} \tag{14}$$

$$\left(\boldsymbol{\psi}^{(k)}\right)^T \mathbf{B}\Delta\boldsymbol{\psi}^{(k)} = 0 \tag{15}$$

Writing Eqs. (14) and (15) in matrix form, we can have

$$\begin{bmatrix} \mathbf{A} - \lambda^{(k)}\mathbf{B} & -\mathbf{B}\boldsymbol{\psi}^{(k)} \\ -\left(\boldsymbol{\psi}^{(k)}\right)^T \mathbf{B} & 0 \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\psi}^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix} \tag{16}$$

If all eigenvalues are distinct, the coefficient matrix is nonsingular. The method using the Newton-Raphson technique, despite its rapid convergence, is not efficient because the new coefficient matrix has to be reformed and refactorized in each iteration step (Lee et al. 1998). This time-consuming procedure could be avoided by applying the modified Newton-Raphson technique as follows

$$\begin{bmatrix} \mathbf{A} - \lambda^{(0)}\mathbf{B} & -\mathbf{B}\boldsymbol{\psi}^{(k)} \\ -\left(\boldsymbol{\psi}^{(k)}\right)^T \mathbf{B} & 0 \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\psi}^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix} \tag{17}$$

The symmetric coefficient matrix of Eq. (17) is also nonsingular. Once the submatrix $\mathbf{A} - \lambda^{(0)}\mathbf{B}$

is decomposed into LDLT (L: lower triangular matrix, D: diagonal matrix), a small number of operations are required to solve Eq. (17), since only the $\mathbf{B}\psi^{(k)}$ in the coefficient matrix is changed in each iteration. However, due to negligence of the small nonlinear term $(\lambda^{(k+1)}-\lambda^{(0)})\mathbf{B}\Delta\psi^{(k)}$, the convergence is lower. Therefore, the improvement of the convergence of the method is needed to apply to a large-scale system.

2.3 Proposed method

To improve the convergence of the method, a step length is introduced in this study as follows

$$\psi^{(k+1)} = \psi^{(k)} + \alpha^{(k)}\Delta\psi^{(k)} \tag{18}$$

Because Eq. (18) is introduced instead of Eq. (9), the residual vector might not be a null vector as in Eq. (13). To minimize the norm of residual vector, the least square technique is used as follows

$$\frac{\partial}{\partial\alpha^{(k)}}\left(\left(\bar{\mathbf{r}}^{(k+1)}\right)^T\bar{\mathbf{r}}^{(k+1)}\right)=0 \tag{19}$$

where

$$\bar{\mathbf{r}}^{(k+1)} = \left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)\left(\psi^{(k)}+\alpha^{(k)}\Delta\psi^{(k)}\right) \tag{20}$$

Solving Eq. (19) for $\alpha^{(k)}$ to yield

$$\alpha^{(k)} = -\frac{\left(\Delta\psi^{(k)}\right)^T\left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^2\psi^{(k)}}{\left(\Delta\psi^{(k)}\right)^T\left(\mathbf{A}-\lambda^{(k+1)}\mathbf{B}\right)^2\Delta\psi^{(k)}} \tag{21}$$

According to our experience, it would be better not to apply $\alpha^{(k)}$ in the first step. The error norm (Bathe 1996) as defined in Eq. (22) after the first step is used to determined whether to apply $\alpha^{(k)}$ from the second step on. Thus, we introduce a checking number γ. If the error norm after the first step is greater than γ, step length is applied and vice versa. The algorithm of the proposed method is shown in Table 1.

Table 1. Algorithm of the proposed method

2.4 Starting values

Initial values of the proposed method can be obtained from the intermediate results of the iteration methods (Gupta 1974; Utku and Clement 1984; Chen et al 1986; Leung 1995) or from the results of the approximate methods (Parlett et al 1985; Kim and Craig 1988; Rajakumar and Roger 1991; Rajakumar 1993; Chen and Taylor 1988; Chen 1994). In this paper the starting values are taken using the symmetric Lanczos method (Chen and Taylor 1988) with a selective reorthogonalization process (Parlett and Scott 1979; Simon 1984), because the method does not need complex arithmetic in the Lanczos recursive process and effectively produces

good approximate values of the systems. If the lower p eigenvalues and the corresponding eigenvectors are desired, the eigenvalue problem reduced by $2p$ Lanczos vectors is solved, and then the p approximate eigenpairs are improved by the proposed method until the predetermined error norm is satisfied.

3. NUMERICAL EXAMPLE

A cantilever beam with lumped viscous-dampers shown in Fig. 1 is analyzed to verify the efficiency of the proposed method. It is modeled by 100 equal elements and has 200 degrees of freedom. The parameters of the system are as follows: the Young modulus E = $2*10^{11}$ N/m2. The inertia of the cross section I = $2.25*10^{-8}$ m4. The cross section area A = $3*10^{-4}$ m2. The density of the material = 8000 kgf/m3. The mass matrix **M** is a consistent one. The damping matrix **C** consists of the Rayleigh damping and the damping contributed from the concentrated dampers. The Rayleigh damping is assumed for the structure itself as

$$\mathbf{C}^{str} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{23}$$

where the coefficients α is 0.002 and β is $2.04*10^{-7}$. On the other hand, the concentrated damping coefficient at each node is assumed to be 0.1. The structures are analyzed using two different methods: the method proposed by Lee et al. (1998) and the proposed method. The error norm (Bathe 1996) is computed by

$$error\ norm = \frac{\left\|\mathbf{A}\psi^{(k)} - \lambda^{(k)}\mathbf{B}\psi^{(k)}\right\|_2}{\left\|\mathbf{A}\psi^{(k)}\right\|_2} \tag{22}$$

The error norm is compared with the predescribed error limit of 10^{-6}.

Fig. 1. Cantilever beam with lumped dampers

Letting the solution time to have 20 eigenpairs with the error norm of 10^{-6} using the proposed method be 1, then the method proposed by Lee et al. (1998) takes 1.08 times.

For each solution method, the convergence of the 14th and the 17th eigenpairs to which is applied is depicted in Fig. 2. As shown in the figure, the convergence of the proposed method is superior to that of Lee et al. (1998).

Fig. 2. Convergence of the 14th eigenpairs (left) and the 17th eigenpairs (right)

4. Conclusions

An efficient method for solving eigenproblems of the non-proportionally damped structures by applying the step length and introducing the orthonormal side condition is presented.

Characteristics of the proposed method illustrated by the numerical results are identified as follows

(1) The convergence rate of the proposed method is improved by introducing the step length.
(2) Therefore, if the exact eigenvalues of the system are known, the proposed method can effectively calculate the corresponding eigenvectors.
(3) The efficiency of the proposed method depends on the checking number γ. Further study on the value of γ is being conducted.

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Table 1. Algorithm of the proposed method

1. Calculate initial eigenvalue $\lambda^{(0)}$ and eigenvectors $\psi^{(0)}$.
2. Iterate the following procedure for each eigenpair:
   (a) $k = 0$
   (b) Check whether to apply the step length (*apply* = Yes/No)
   (c) For $k = 1$
   (d) Define $\begin{bmatrix} \mathbf{A}-\lambda^{(0)}\mathbf{B} & -\mathbf{B}\psi^{(k)} \\ -(\psi^{(k)})^T\mathbf{B} & 0 \end{bmatrix}$
   (e) Define $-\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix}$ where $\mathbf{r}^{(k)} = \mathbf{A}\psi^{(k)} - \lambda^{(k)}\mathbf{B}\psi^{(k)}$
   (f) Solve $\begin{bmatrix} \mathbf{A}-\lambda^{(0)}\mathbf{B} & -\mathbf{B}\psi^{(k)} \\ -(\psi^{(k)})^T\mathbf{B} & 0 \end{bmatrix}\begin{Bmatrix} \Delta\psi^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{r}^{(k)} \\ 0 \end{Bmatrix}$ for $\begin{Bmatrix} \Delta\psi^{(k)} \\ \Delta\lambda^{(k)} \end{Bmatrix}$
   (g) Compute $\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)}$

   If *apply* = Yes $\alpha^{(k)} = -\dfrac{(\Delta\psi^{(k)})^T(\mathbf{A}-\lambda^{(k+1)}\mathbf{B})^2\psi^{(k)}}{(\Delta\psi^{(k)})^T(\mathbf{A}-\lambda^{(k+1)}\mathbf{B})^2\Delta\psi^{(k)}}$

   else $\alpha^{(k)} = 1$; end

   $\psi^{(k+1)} = \psi^{(k)} + \alpha^{(k)}\Delta\psi^{(k)}$

   $\mathbf{r}^{(k+1)} = \mathbf{A}\psi^{(k+1)} - \lambda^{(k+1)}\mathbf{B}\psi^{(k+1)}$ and $error\ norm = \dfrac{\|\mathbf{r}^{(k+1)}\|_2}{\|\mathbf{A}\psi^{(k+1)}\|_2}$

   (h) Update $\mathbf{A}-\lambda^{(0)}\mathbf{B}$ in (d) to $\mathbf{A}-\lambda^{(k)}\mathbf{B}$ if necessary to improve convergence
   (i) If *error norm*>predetermined error limit, go to (c) with $k = k + 1$

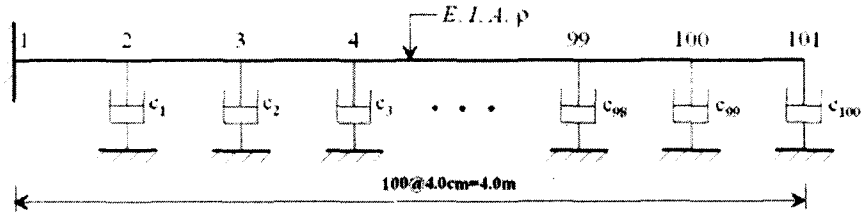

Fig. 1. Cantilever beam with lumped dampers

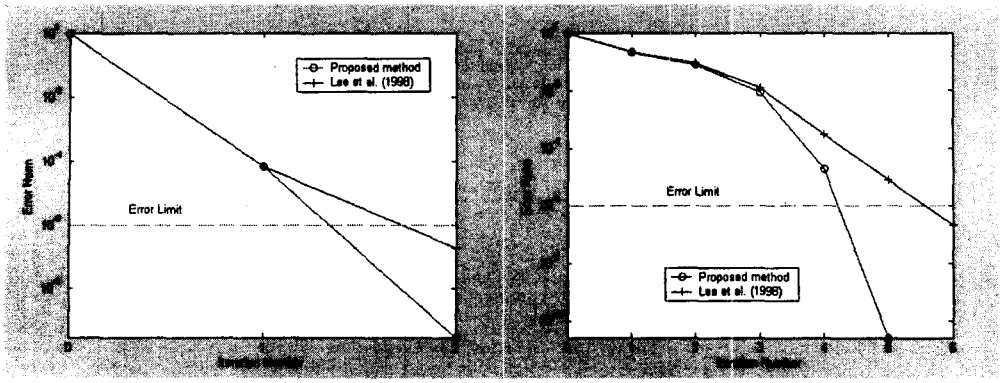

Fig. 2. Convergence of the 14th eigenpairs (left) and the 17th eigenpairs (right)