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Probabilistic stability analysis of underground structure using stochastic finite element method

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Abstract: It can be said that rock mass properties are characterized not by a mean value but by values with variation due to its characteristic uncertainty. This characteristic is one of the most important parts for the design of underground structures, but yet to be fully examined. Stochastic finite element method (SFEM) has been developed in order to take the randomness of structural systems into account. Using SFEM, the response variability of structural system can be obtained and it leads probabilistic stability of structure to be analyzed. In this study, displacements response variability of circular opening with hydrostatic stress field are analyzed in terms of rock mass properties having a certain mean and a standard deviation using the SFEM. The analyzed response variability shows that the necessity of probabilistic stability analysis of underground structures using reliable mean value and standard deviation of deformation modulus.

1. Introduction

Mechanical behaviors in engineering problems are often studied by using numerical analyses. Among a number of numerical analysis methods, finite element analysis (FEA) has widely been adopted, and showed excellent applicability. However, it is necessary in a deterministic FEA to determine a priori geometry and material properties of a model to be simulated, which are fixed during the analysis. In rock engineering, most of safety analyses have been treated by the deterministic FEA. However, it cannot account for spatial variability of rock mass properties. In addition, the mean value of rock mass properties, which are obtained by various in-situ and laboratory tests, is insufficient to represent in-situ rock mass characteristics due to spatial variability. Therefore, stability analyses with the mean value are possibly far beyond actual structural behaviour of underground structures.

The stochastic FEA is contrary to the deterministic FEA in its concept as the former has been developed in order to take the randomness of structural system into account. This method is based on an assumption that the randomness are in probability. The randomness inherently existing in structural system of rock mass affects structural response. Therefore, the effect can be taken into account by a number of statistical properties such as mean, standard deviation, and so on, which represent statistics of response variability. Deformation modulus having spatial variability is one of most important variables required in stability analysis (Ra, 1999, Kim and Gao, 1995). In this study response variability of displacement taking place on circular opening in hydrostatic stresses is analyzed by using SFEM which considers randomness of deformation modulus having a certain mean and standard deviations.

2. Stochastic Finite Element Method

In SFEM, the randomness of structural system can be taken into account by Monte Carlo simulation, perturbation method, reliability based method, weighted integral method, and so on. In this study, the weighted integral method is adopted as it allows a relatively large variability of the random variable and uses the same mesh as used in deterministic FEA (Matthies et al., 1997, Deodatis, 1991).

Weighted integral method

Weighted integral method is a kind of local average method in view of discretization of the random field parameter. However it dose not require a separate discretization of the random field. Therefore, there is no independent mesh for the stochastic part. The mathematical representation in given by

$$X = \int_{\Omega} f(x)\ g(x)\ d\Omega \tag{1}$$

where f(x) represents random field and g(x) is a deterministic function.
The weighted integral in Eq. (1) represents random variable X in domain Ω.

The usability of Eq. (1) in stochastic analysis comes from the fact that the expected value operator and covariance operator are easy to be calculated. Substituting the expected value operator and covariance operator for Eq. (1) becomes as follows:

$$E[X] = X_0 = \int E[f(x)]g(x)d\Omega = 0 \tag{2}$$

$$Cov[X_1, X_2] = E[(X_1 - X_{1_0})(X_2 - X_{2_0})] = \int_{\Omega_1}\int_{\Omega_2} E[f(x_1)f(x_2)]g(x_1)g(x_2)d\Omega_2 d\Omega_1 \tag{3}$$

where E[ ] is expected value operator and Cov[,] is covariance matrix, x_1 and x_2 are random variables, and subscribe '0' represents mean value. In Eq. (3), expected value operator is solved by definition of auto-correlation function.

Random field of deformation modulus

Assuming that deformation modulus of each element in finite element model randomly varies over its area, and it can be formulated as follows:

$$E(x) = E_0[1 + f(x)] \tag{4}$$

where E_0 is a mean value of deformation moduli and f(x) is one-dimensional homogeneous stochastic field with zero mean value.

Auto-correlation function

Auto-correlation function relates to relative position of random variable. The mathematical form is as follows:

$$R_{aa}(\xi) = E[a(x)a(x + \xi)] \tag{5}$$

where $R_{aa}(\xi)$ is a one-dimensional auto-correlation function, a is a random variable, x is a position vector, and ξ is a relative position vector. In Eq. (5), the auto-correlation function is only a function of ξ.

The auto-correlation function used in this study is identical to that used by Deodatis et al. (1991) and given by

$$R_{ff}(\xi, \eta) = \sigma_{ff}^{\ 2} \cdot \exp\{-\frac{|\xi| + |\eta|}{d}\} \tag{6}$$

where $R_{ff}(\xi, \eta)$ is a two-dimensional auto-correlation function, σ_{ff} is a coefficient of variation (COV) of f(x), and ξ and η are vectors of relative distance x and y direction, respectively. Correlation distance, d, of random variable represents decay of correlation with relative distance.

When all finite elements for the given domain are characterized by the same stochastic field, the expected value operator appearing in Eq. (3) is calculated as following form:

$$E[f(x_1)f(x_2)] = R_{ff}(x_2 - x_1) \tag{7}$$

Stochastic element stiffness matrix

In deterministic FEA, element stiffness matrix can be written as following form:

$$K^e = \int_{\Omega_e} B_e^{\ T} D_o B_e d\Omega_e \tag{8}$$

where K is stiffness matrix, B is gradient matrix, D_o is constitutive matrix, superscript T is transposed matrix, and subscript e is element.

For the stochastic FEA, D_o, in Eq. (8), would be replaced by D_s, which is obtained by inserting material properties in D_o into Eq. (4). In this way, the element stiffness matrix can be divided into deterministic part and stochastic part as following form:

$$K^e = K_0 + \Delta K^e = \int_{\Omega_e} B_e^{\ T} D_o B_e d\Omega_e + \int_{\Omega_e} f(x) B_e^{\ T} D_o B_e d\Omega_e \tag{9}$$

Response variability of displacement

In order to calculate response variability of displacement, first-order Taylor series expansion of displacement vector U is performed. Reduced equation using the equilibrium equation is as follows:

$$U \cong U_0 - \sum_{e=1}^{N_e} \sum_{wi=1}^{N_{wi}} X_{wi}^e K_0^{-1} \left(\frac{\partial K}{\partial X_{wi}^e} \right)_E U_0 \tag{10}$$

where U_0 is the mean value of displacements, X_{wi} is the random variable which is obtained by weighted integral, N_e is the total number of elements, N_{wi} is the total number of weighted integrals, subscript E represents expected value, e is element number, wi means weighted integral, and superscript '-1' represents the inverse matrix.
Using Eq. (10), the expected value and the covariance of displacement vector can be obtained. The expected value of displacement is same as the deterministic FEA result, and its mathematical form is given by

$$E[U] = U_0 \tag{11}$$

The covariance of displacement is obtained by using its definition as

$$\mathrm{Cov}[U,U] = E[(U-U_0)(U-U_0)^T] = \sum_{e_1}^{N_e} \sum_{e_2}^{N_e} K_0^{-1} E[\Delta K^{e_1} U_0 U_0^T \Delta K^{e_2}] K_0^{-T} \tag{12}$$

where e_1 and e_2 are element number, and superscript '-T' is transposed matrix of inverse matrix. The expected value operator in Eq. (12) can be rewritten by using auto-correlation function to give

$$E[\Delta K^{e_1} U_0 U_0^T \Delta K^{e_2}] = \int_{\Omega_1} \int_{\Omega_2} R_{ff}(\xi,\eta) B_{e_1}^T D_0^e B_{e_1} U_0 U_0^T B_{e_2}^T D_0^e B_{e_2} d\Omega_1 d\Omega_2 \tag{13}$$

When calculating Eq. (13), it is noted that ξ and η have no '0' values but a certain value with relative position in the element.
In two-dimension problem, mean value, E[z], variance, Var(z), of nodal displacement can be obtained as following forms:

$$E[z] = \sqrt{E[x]^2 + E[y]^2} \tag{14}$$

$$\mathrm{Var}(z) = E[x]^2 + E[y]^2 + \mathrm{Var}(x) + \mathrm{Var}(y) - E[z]^2 = \mathrm{Var}(x) + \mathrm{Var}(y) \tag{15}$$

where E[x] is the expected value and Var(x) is the variance of the nodal displacement in x-direction, E[y] is the expected value and Var(y) is the variance of the nodal displacement in y-direction.

3. Numerical analysis

Analysis description

Stochastic FEA was performed for a circular opening of 5m radius at hydrostatic stress field. Deformation modulus is regarded as a random variable. According to Kim and Gao (1995), the COV of deformation modulus of rock mass was investigated up to 50%. In this study, the mean value of deformation modulus equals 20GPa (corresponding to RMR 60) and the COV of deformation modulus varies 10%, 20%, 30% in the analysis. According to Ra (1999), the correlation distance of deformation modulus of rock mass was about between 3 and 8m. In order to regard small and large values of correlation distance, the correlation distance of deformation modulus varies minimum 0.1m and maximum 300m which is 10 times of analysis range of this study.
Finite element mesh is shown in Fig. 1. The analysis range is 30m x 30m and the right and left sides are fixed in displacement of horizontal direction and the upper and lower sides are fixed in displacement of vertical direction.

Analysis of response variability of displacement

The displacement of 1.46mm at point A is obtained by deterministic FEA (Fig. 1). The variance of the COV of displacement at point A in terms of correlation distance and the COV of deformation modulus is plotted in Fig. 2. It shows that the COV of displacement increase with increasing correlation distance. When the correlation distance is over about 100m, the COV of displacement tends to converge. The COV of displacement increases linearly with the COV of deformation modulus and non-linearly with correlation distance.

With varying correlation distance, the COV of displacement ranges from 0.4% to 10%, from 0.9% to 20%, from 1.3% to 30% according to the COV of deformation modulus equals 10%, 20%, 30%, respectively. This result shows that the displacement obtained by deterministic FEA deviates largely with respect to the COV and the correlation distance of deformation modulus

Assuming response variability of displacement is normal distribution, Table 1 summarizes maximum displacement in reliability concept. The ratio of maximum displacement to expected value of displacement with the COV of deformation modulus at correlation distance is 300m. At 99% confidence level, the ratio is 1.23, 1.47, 1.70 with the COV of deformation modulus equals 10%, 20%, 30%. Table 2 summarizes the ratio at the correlation distance is 5m, where the ratio is up to 1.42. This result is important for stability analysis of underground structure but cannot be obtained by deterministic FEA.

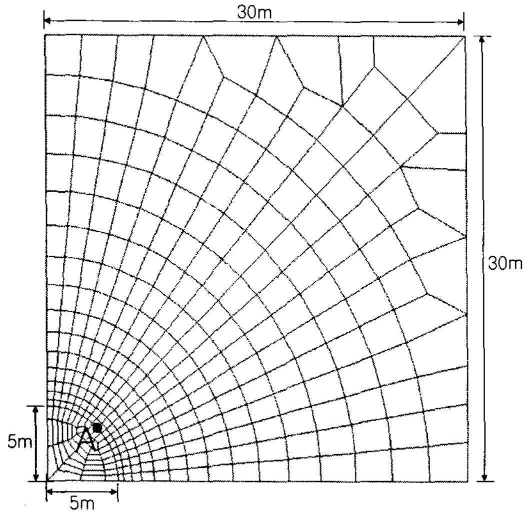

Fig. 1. Finite element mesh for stochastic FEA.

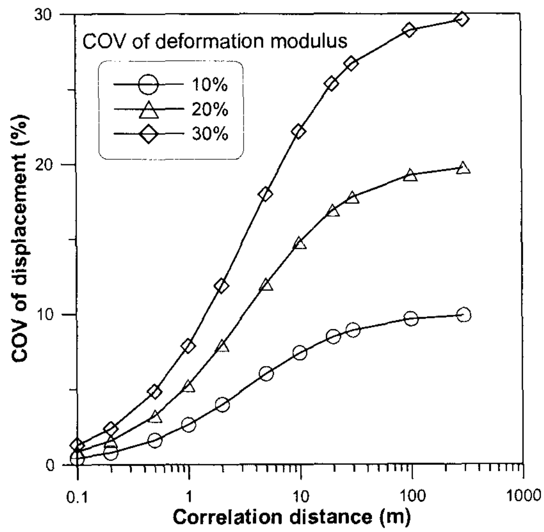

Fig. 2. The variation of the COV of displacement at point A with increasing correlation distance for three different COVs of deformation modulus.

Table 1. The ratio of maximum displacement to expected value of displacement with varying the COV of deformation modulus and confidence level at correlation distance is 300m.

Confidence level / COV of deformation modulus	90%	95%	97.5%	99%
10%	1.13	1.17	1.20	1.23
20%	1.26	1.33	1.39	1.47
30%	1.39	1.50	1.59	1.70

Table 2. The ratio of maximum displacement to expected value of displacement with varying the COV of deformation modulus and confidence level at correlation distance is 5m.

Confidence level / COV of deformation modulus	90%	95%	97.5%	99%
10%	1.08	1.10	1.12	1.14
20%	1.16	1.20	1.24	1.28
30%	1.23	1.30	1.35	1.42

In the FEA, result accuracy is dependent on mesh refinement. It is noteworthy that weighted integral method dose not require an additional mesh for stochastic analysis. Stochastic analysis uses the same mesh as used in deterministic FEA. In order to analyze the effect of mesh refinement, number of elements around the quarter circular opening varying 4, 6, 10, and 18 in the same condition of above analysis. Fig. 3 shows the effect of mesh refinement with the COV equals 10% and correlation distance is 5m of deformation modulus. The number of elements is up to 6, the COV of displacement at point A dose not change significantly. It is observed that mesh refinement effect is negligible when using weighted integral method for stochastic FEA.

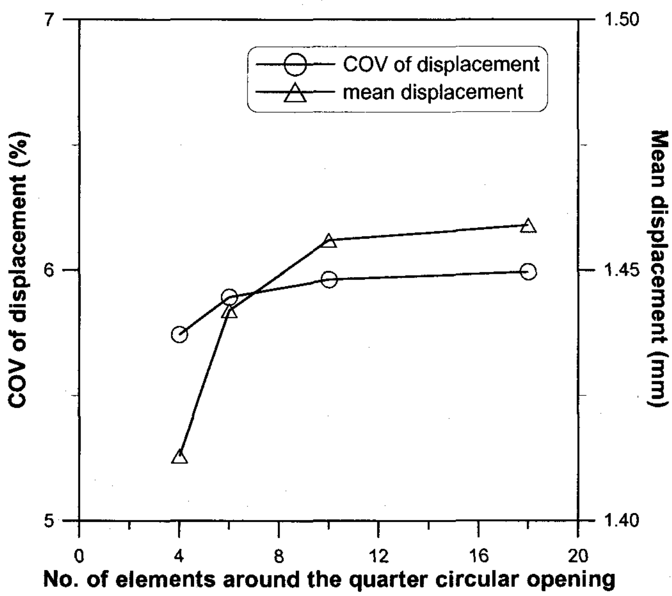

Fig. 3. The effects of mesh refinement on the COV of displacement and mean displacement.

3. Conclusions

In this study, the stochastic finite element program named stochastic Utah2 (SU2) has been developed for analysis of displacement of underground structure. The weighted integral method adopted in order to take into account randomness of structural system of underground structure. Numerical analysis for mesh refinement effect shows the advantage of weighted integral method, dose not require addition mesh, for stochastic FEA. Numerical examples, using SU2, have been performed for deformation modulus with random variable of a mean value and standard deviations at hydrostatic stress field.

The COV of deformation modulus greatly affects to the response variability of displacement. Correlation distance of deformation modulus is also the important variable to affect the response variability of displacement.

Assuming the response variability of displacement, the result of stochastic FEA, to be of normal distribution, the ratio of the maximum displacement to the expected value of displacement (deterministic FEA result) is 1.70 with the COV and correlation distance of deformation modulus equal 30% and 300m at 99% confidence level. Also, with the correlation distance of deformation modulus equals 5m, the ratio is 1.42 at 99% confidence level.

In the deterministic FEA, only the expected value of displacement is obtained. However, using stochastic FEA, the response variability of displacement can be obtained. Using this result, it is possible to perform reliability analysis of underground structure with confidence level. Furthermore, it is concluded that the analysis of the failure probability needs to be performed by using failure criterion if response variability of stresses is analyzed.

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