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Effects of Dilatation and Vortex Stretching on Turbulence in One-Dimensional and Axisymmetric Flows

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일차 및 축대칭유동에서 밀도변화가 난류에 미치는 영향

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An analytic approach is attempted to predict the amplification of turbulence in compressible flows experiencing one-dimensional and axisymmetric bulk dilatation. The variations of vortex radius and vorticity are calculated, and then the amplification of turbulence is obtained from them by tracking three representative vortices. For a one-dimensionally compressed flow, the present analysis slightly underestimates the amplification of velocity fluctuations and turbulent kinetic energy, relative to that of rapid distortion theory in the solenoidal limit. For an axisymmetrically distorted flow, the amplification of velocity fluctuations and turbulent kinetic energy depend not only on the density ratio but also on the ratio of streamwise mean velocities, which represents streamwise vortex contraction/stretching. In all flows considered, the amplification of turbulence is dictated by the mean density ratio. In the axisymmetric flow, streamwise vortex stretching/contraction, however, alters the amplification slightly.

1. Introduction

Since 1930s, several theoretical attempts have been carried out to predict the amplification of turbulence in a flow experiencing contraction in the contraction of a wind tunnel or in a flow experiencing dilatational distortion. Using the conservation of energy and angular momentum of a rotating cylindrical fluid element, Prandtl [1933] predicted the amplification of streamwise and spanwise turbulent velocities in a flow passing through the contraction of a wind tunnel. Taylor [1935] also tried to calculate the turbulence amplification by predicting vorticity amplification using a theory based on the conservation of circulation for an inviscid fluid.

For a low/weak level of turbulence, Ribner and Tucker [1953] and Ribner [1954] predicted the amplification of turbulent velocities using linearized equations where dissipation was neglected. In this analysis, the velocity fluctuation was treated as a wave, as in Moore [1954]. As will be discussed later in detail, the result of LIA is identical to that of Rapid Distortion Theory (RDT), which is a more popular theoretical approach, in the solenoidal limit.

The basic concept of RDT was proposed by Prandtl [1933] and Taylor [1935], where turbulence distortion/deformation time scale is sufficiently smaller than the time-scales of fluctuations. For the two extreme cases of solenoidal and dilatational modes, the amplification of Turbulent Kinetic Energy (TKE) was analytically predicted by RDT for flows with a finite turbulent Mach number [Cambon et al. 1993; Jacquin et al. 1993]. Simone et al. [1997] used the RDT in a flow experiencing one-dimensional, isotropic compression and/or pure shear. The RDT was also used in two-dimensional supersonic turbulent boundary layer experiencing a rapid expansion by Jayaram et al. [1989]. Their analysis, however, was only valid immediately downstream of the last expansion fan.

Another theoretical approach is to track the behavior of representative vortices to predict the amplification of turbulence. In this analysis, the vortex tracking is conducted to evaluate the variations of the radius and vorticity of the representative vortices, which are later used to calculate the amplification of turbulence. Basically, this approach is similar to that of Prandtl [1933]. The effect of the mean density variation was taken into account in the approach, while it was not in the analysis of Prandtl. By using the conservation of angular momentum of a vortex as Prandtl [1933] did in incompressible flows, the amplification of velocity fluctuations in the streamwise and normal directions was successfully predicted for a supersonic boundary layer experiencing compression and expansion [Kim et al. 2001]. Their analysis for the streamwise vorticity clearly shows the effect of vortex stretching/contraction and/or dilatation on vorticity. The effect of dilatation on vorticity was much greater than that of streamwise vortex stretching or contraction. The term vortex contraction refers to negative stretching or shortening of a vortex tube in the present paper. The present study is to further Kim et al.'s [2001] research and to provide physical insight into the mechanism of the amplification of turbulence and TKE in flows experiencing axial and axisymmetric dilatation.

2. Analysis

The present analysis tracks three representative vortices to evaluate the variations in the radius and vorticity of the three vortices. Once these two quantities are obtained for each vortex, the amplification of turbulent velocities and TKE is calculated from them. In this sense, this analysis can be called "Vortex Tracking Analysis." The velocity

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fluctuation in one direction is caused by the vortices normal to the direction. For example, the streamwise velocity fluctuation is induced by cross-streamwise vortices. Thus, by tracking three representative vortices aligned with three corresponding directions of the coordinate system, the amplification of turbulent quantities can be obtained. Since the amplification of turbulence is calculated from the variation in vorticity of three vortices in this analysis, the interaction between vorticies is not taken into account as in RDT. Also the effects of viscosity are neglected in the analysis.

The two conservation equations of mass and angular momentum are used to calculate the variation of the radius and vorticity of a given vortex. The mass conservation law is applied first to evaluate the radius after the dilatational distortion. Then the vorticity variation due to the distortion is calculated by using the conservation law of the angular momentum of the vortex filament as in Kim et al. [2001]. When compared to the RDT, this analysis is relatively simple and straightforward. However, this analysis helps to figure out the roles of the mean flow parameters, for example the mean density and vortex stretching, in turbulence and anisotropy.

2.1 Axial Compression/Expansion

Amplification of vorticity

When a vortex in a flow experiences axial or one-dimensional dilatation, its dimensions in cross-stream directions remain the same as depicted in Fig. 1. Thus an equation of the conservation of mass for a streamwise vortex filament is written as:

$$\rho_1 L_1^S \left(R_1^C\right)^2 = \rho_2 L_2^S \left(R_2^C\right)^2, \tag{1}$$

where ρ, L, and R are respectively the mean density and the length and radius of the vortex. The superscripts S and C indicate streamwise and cross-streamwise, respectively. Since the dimensions in directions normal to the dilatation direction are kept the same, the radius of the vortex before the distortion equals that after the distortion, i.e., $R_1^S = R_2^S$. From this and Eq. 1, one can show that

$$\rho_1 L_1^S = \rho_2 L_2^S. \tag{2}$$

Considering a continuity equation of the mean flow

$$\rho_1 U_1^S S_1 = \rho_2 U_2^S S_2, \tag{3}$$

L_2^S / L_1^S is equal to U_2^S / U_1^S since the cross-sectional area of the flow boundary S is constant for the one-dimensional flow. As will be shown later, the relation, $L_2^S / L_1^S = U_2^S / U_1^S$, holds also for the streamwise vortex in two-dimensional flows.

The conservation of angular momentum of a vortex filament with radius R says that

$$I_1 \Omega_1^S = I_2 \Omega_2^S, \tag{4}$$

where I is the moment of initia of the vortex filament, $I = mR^2/2$. Since mass m is conserved, the equation is reduced to

$$\left(R_1^S\right)^2 \Omega_1^S = \left(R_2^S\right)^2 \Omega_2^S \tag{5}$$

From $R_1^S = R_2^S$ for the streamwise vortex in a one-dimensional flow, one can show that streamwise vorticity does not vary for axial dilatation:

$$\frac{\Omega_2^S}{\Omega_1^S} = \left(\frac{R_1^S}{R_2^S}\right)^2 = 1. \tag{6}$$

These relations can be used for axially compressed and expanded flows.

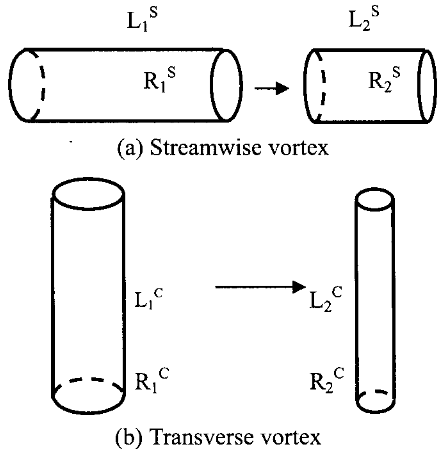

Fig. 1 Schematic of vortices experiencing axial or one-dimensional compression.

For the cross-streamwise vortex, the equation of mass conservation is

$$\rho_1 \left(R_1^C\right)^2 L_1^C = \rho_2 \left(R_2^C\right)^2 L_2^C. \tag{7}$$

The length of the transverse vortex filament remains unaltered for the axially compressed/expanded case, i.e., $L_1^C = L_2^C$, and thus the ratio of radii is

$$\frac{R_2^C}{R_1^C} = \left(\frac{\rho_1}{\rho_2}\right)^{1/2} = \frac{1}{\sqrt{k_d}}, \tag{8}$$

where $k_d = \rho_2/\rho_1$. By using this equation and the angular momentum conservation equation $\left(R_1^C\right)^2 \Omega_1^C = \left(R_2^C\right)^2 \Omega_2^C$, the amplification of cross-stramwise vorticity is

$$\frac{\Omega_2^C}{\Omega_1^C} = \left(\frac{R_1^C}{R_2^C}\right)^2 = k_d. \tag{9}$$

As shown in this section, the three conservation equations, Eqs. 1-3, are all required in the present analysis.

Amplification of velocity fluctuations

The streamwise and cross-streamwise rms fluctuations σ^S and σ^C are respectively written as [Kim et al. 2001]:

$$\sigma^S = \sqrt{\frac{(\Omega^C R^C)^2 + (\Omega^C R^C)^2}{2}}, \tag{10}$$

$$\sigma^C = \sqrt{\frac{(\Omega^S R^S)^2 + (\Omega^C R^C)^2}{2}}. \tag{11}$$

By using Eq. 10, the streamwise velocity fluctuation σ_2^S after expansion/compression is:

$$\sigma_2^S = \Omega_2^C R_2^C. \tag{12}$$

Substituting Eqs. 8 and 9 into the equation, one obtains

$$\sigma_2^S = \sqrt{k_d}\left(\Omega_1^C R_1^C\right).$$

From this equation, the amplification of rms streamwise velocity fluctuation $\alpha^S \equiv \sigma_2^S / \sigma_1^S$ is

$$\alpha^S = \sqrt{k_d}, \tag{13}$$

where $\sigma_1^S = \Omega_1^C R_1^C$.

Similarly, the rms of cross-streamwise velocity fluctuation $\sigma_2{}^C$ is

$$\sigma_2{}^C = \sqrt{\frac{(\Omega_2^S R_2^S)^2 + (\Omega_2^C R_2^C)^2}{2}}.$$

Substituting Eqs. 6, 8, and 9 into the equation and using $R_1{}^S = R_2{}^S$, one can have

$$\sigma_2{}^C = \sqrt{\frac{(\sigma_1{}^S)^2 + k_d(\sigma_1{}^C)^2}{2}}.$$

For initially isotropic turbulence, one can set $\sigma_{10} = \sigma_1{}^S = R_1^S\Omega_1^S = \sigma_1{}^C = R_1^C\Omega_1^C$, as Kim et al. [2001] did, and thus one obtains the amplification of rms of the cross-streamwise velocity fluctuation $\alpha^C \equiv \sigma_2{}^C/\sigma_{10}$ as:

$$\alpha^C = \sqrt{\frac{1+k_d}{2}}. \tag{14}$$

The amplification of velocity fluctuation in the streamwise direction α^S is greater than that in the cross-streamwise direction α^C when the flow is compressed as shown in Eqs. 13 and 14, since k_d is greater than 1.

The amplification of TKE α^{TKE} is calculated by using Eqs. 13 and 14 as:

$$\alpha^{TKE} = \frac{1+2k_d}{3}. \tag{15}$$

As the density ratio increases by compression, TKE is amplified in propotion to density ratio k_d monotonically. For expansion cases, the amplification factor approaches 1/3 as the density ratio k_d approaches zero. A more detailed disscussion will follow.

2.2 Axisymmetric Flow

By using the conservation equations of Eqs. 1, 3, and 4, one can get the amplification factors of the streamwise and cross-streamwise velocity fluctuations and TKE in an axisymmetric flow. The procedure the almost the same as in the previous section, and thus only the resultant equations will be given.

The amplification factors of rms streamwise and cross-streamwise turbulent velocity $\alpha^S \equiv \sigma_2{}^S/\sigma_{10}$ and $\alpha^C \equiv \sigma_2{}^C/\sigma_{10}$ are resectively,

$$\alpha^S = \left(\frac{k_d}{k_U}\right)^{1/4}, \tag{16}$$

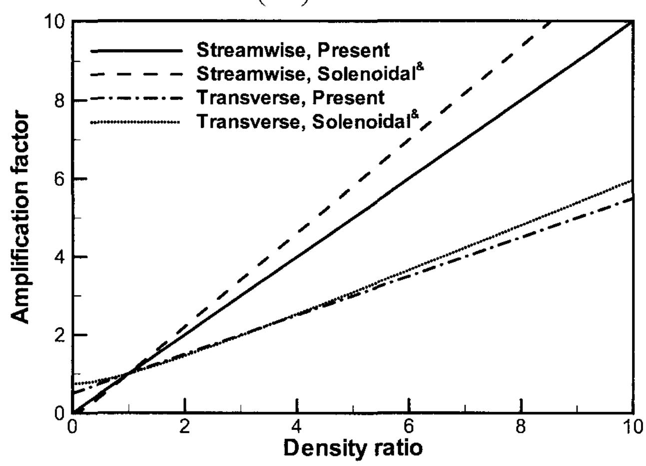

Fig. 2 Amplification factors of velocity fluctuations, square of α^S and/or α^C, with an increasing density ratio k_d in a one-dimensional flow: compression for $k_d>1$ and expansion for $k_d<1$. The curves for the solenoidal limit (&) were taken from Ribner and Tucker [1953].

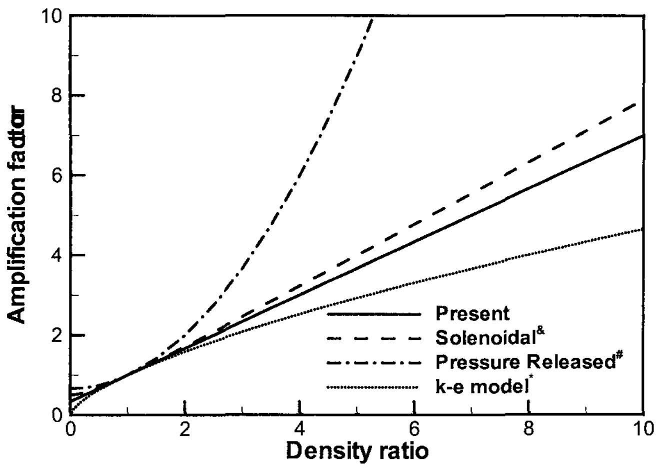

Fig. 3 Amplification factors of TKE with an increasing density ratio k_d in a one-dimensional flow: compression for $k_d>1$ and expansion for $k_d<1$. The related references for the solenoidal limit (&), pressure limit (#), and k-ε model (*) are respectively Ribner and Tucker [1953], Cambon et al. [1993], and Gillet et al. [1998].

$$\alpha^C = \sqrt{\frac{(k_d/k_U)^{1/2} + k_d k_U}{2}}, \tag{17}$$

where $k_U = U_2/U_1$. From thses the amplification factor of TKE is obtained as:

$$\alpha^{TKE} = \frac{k_d k_U + 2\sqrt{k_d/k_U}}{3}, \tag{18}$$

for initially isotropic turbulence.

3. Results and Discussion

3.1 Axial or one-dimensional compression

An axial compression/expansion occurs in piston engines and shock tubes. For these cases, the dimensions of the flow boundary and vortices aligned in the transverse directions remain fixed. The amplification of velocity fluctuations, the square of α^S and/or α^C, is shown in Fig. 2 for axially compressed and expanded flows. When the present prediction curve is compared with that of RDT or LIA, it underestimates velocity fluctuations in both the streamwise and transverse directions. If air is compressed, by a shock wave, the maximum density ratio is 6. For a density ratio less than 6, the amplification predicted by the present analysis agrees well with that of Ribner and Tucker [1953] or of the solenoidal limit of RDT. In the streamwise direction, the amonut of underprediction is greater than that in the cross-streamwise direction.

The amplification of turbulent kinetic energy with increasing density ratio is shown in Fig. 4. Also shown in the figure are the solenoidal and pressure-released limits obtained from RDT [Cambon et al. 1993, Jacquin et al. 1993]. The equation of the solenoidal limit was derived by RDT,

$$\alpha^{TKE} = \frac{1}{2}\left(1 + k_d{}^2 \frac{\tan^{-1}\sqrt{k_d{}^2 - 1}}{\sqrt{k_d{}^2 - 1}}\right), \tag{19}$$

which appears in Durbin and Zeman [1992], Cambon et al. [1993], Jacquin et al. [1993], and Simone et al. [1997]. The equation can be easily obtained from the amplification factors of turbulent velocities in

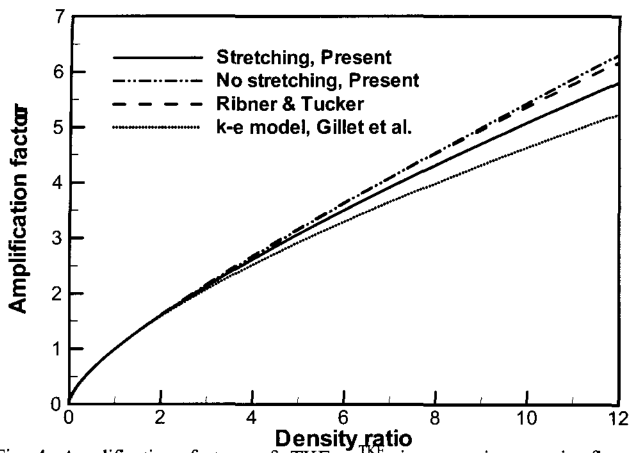

Fig 4 Amplification factors of TKE α^{TKE} in an axisymmetric flow: compression for k_d>1 and expansion for k_d<1. The upstream Mach numbers are 1 and 5 for expansion and compression cases, respectively

Ribner and Tucker [1953]. When the distortion Mach number is significantly greater than unity ($M_d >> 1$), the amplification is limited by the pressure-released limit where pressure fluctuations are neglected [Cambon et al. 1993, Jacquin et al. 1993]. This limit is also derived using the RDT and is written as:

$$\alpha^{TKE} = \frac{2 + k_d^{\ 2}}{3}. \qquad (17)$$

Gillet et al. [1998] also obtained a theoretical equation predicting the amplification of TKE from the k-ε model equations when the flow distortion rate is much faster than the energy cascade rate, i.e., when the conditions of rapid distortion are satisfied. The equation is shown as:

$$\alpha^{TKE} = k_d^{\ 2/3} \qquad (18)$$

These three curves defined by Eqs. 19-21 are depicted in Fig. 4 for comparison. The equations 15, 19, and 21 are obtained under similar conditions: rapid distoriton and negligible dissipation.

The amplification predicted by the present analysis is a little lower than that of the solenoidal limit of RDT and the difference between them is less than 9.4% for a density ratio less than 6, which is the maximum value attainable by an adiabatic normal shock wave in air. The relation, which was obtained from the k-ε model by Gillet et al. [1998], significantly underestimates the amplification of TKE. Thus, the present analysis predicts the amplification of TKE better than that of Gillet et al. [1998]. In this flow, the amplification of TKE is independent of vortex stretching/compression as can be inferred from Eqs. 15 and 21.

3.2 Axisymmetric Compression

In this flow regime, the amplification of TKE depends on the density and streamwise velocity ratios as Eq. 21 says. As shown in Fig. 4, the difference between the present and Ribner and Tucker's prediction curves is very small and even negligible in an engineering sense. For a small upstream Mach number less than 3, it is hard to tell the difference. As the upstream Mach number increases, the prediction curve of Ribner and Tucker approaches that for no-contraction case of the present analysis.

4. Conclusions

A theoretical analysis was carried out to predict the amplification factors of turbulent quantities in flows experiencing one-dimensional and axisymmetric dilatation.

For the one-dimensional flow, the relations predicting the amplification of turbulent velocities and TKE were obtained from the conservation equations of mass and angular momentum of a vortex, which experiences either expansion or compression dilatation. When the distortion Mach number was much smaller than unity, the present analysis slightly underestimated the amplification of turbulent velocities and TKE. For a density ratio less than 6, the agreement between the present analysis and RDT or LIA was quite good. In this flow regime, the amplification factors depended only on the density ratio.

For the axisymmetrically distorted flow, the amplification factors of turbulent velocities and TKE depended not only on the density ratio but also on the ratio of streamwise mean velocities, which represents streamwise vortex contraction/stretching. The present analysis slightly overestimated the streamwise velocity fluctuation and underestimated the transverse velocity fluctuations when compared to Ribner and Tucker's analytic results. However, the agreement in the amplification of TKE was excellent.

In both one-dimensional and axisymmetric flows, the bulk dilatation most probably governed the amplification of turbulent fluctuations for a given flow regime.

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