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An asymptotic analysis of the Taylor-Proudman flow in a rapidly-rotating compressible fluid

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압축성 회전유체에서 발생하는 Taylor-Proudman 유동에 대한 점근해석

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Abstract

A matched asymptotic analysis is conducted for a compressible rotating flow in a cylindrical container when a mechanical and/or a thermal disturbance is imposed on the wall. The system Ekman number is assumed to be very small. The conditions for the Taylor-Proudman column in the interior, which were also given in the companion paper (Park & Hyun, 2002) by means of the energy balancing analysis, have been re-derived. The concept of the variable, the energy content $e[\equiv T+2\,\alpha^2 rv]$, is reformulated, and its effectiveness in characterizing the energy transport mechanism is delineated. It is seen that, under the condition of the Taylor-Proudman column, numerous admissible theoretical solutions for interior flow exist with an associated wall boundary condition. Some canonical examples are illustrated with comprehensive physical descriptions. The differential heating problem on the top and bottom endwall disks is revisited by using the concept of the energy content. The results are shown to be in line with the previous findings.

1. Introduction

The purpose of this paper is to perform a comprehensive and systematic theoretical analysis for the Taylor-Proudman column flow in a rapidly rotating container with a height H^* and a radius $r_0 H^*$. It will be demonstrated that the results of the present theoretical endeavors are consistent with the previously-obtained depictions of compressible rotating flows. In particular, the conditions for the Taylor-Proudman column are reformulated, and a thorough understanding of the physical pictures is acquired. The significance of the energy content e, which was identified in the preceding paper [see Park & Hyun, 2002], is re-established. Details of the flows which are admissible under the conditions of the Taylor-Proudman column are delineated by theoretical means.

The majority of prior studies of compressible rotating flows have been concerned with the technological applications of gas centrifuges. Sakurai & Matsuda (1974), and Nakayama & Usui (1974) analyzed thermally-driven flows of a rapidly-rotating compressible fluid in a cylinder. A temperature contrast was applied between the two rotating endwall disks, and a thermally-conducting cylindrical sidewall was considered. These studies illustrated that the flow features were substantially different from those of an incompressible Boussinesq fluid. Matsuda & Hashimoto (1976) carried out an analysis for the situation when the rotating endwall disks were insulating and the sideall was isothermal. Bark & Hultgren (1979) performed the analysis in an extended parameter space. A review was provided by Conlisk (1985).

Descriptions of the Stewartson layers on the cylindrical sidewall were given by Matsuda & Hashimoto (1978) and by Matsuda & Takeda (1978). Bark & Bark (1976), Hultgren & Bark (1986) and Park & Hyun (1997, 1998) conducted studies on the boundary layer flows when the thickness of the boundary layer is comparable with the density scale height.

The concept of the Taylor-Proudman column occupies a center stage in the discussion of rotating flows of an incompressible fluid. It is anticipated that the Taylor-Proudman column will equally be a pivotal dynamical element in rotating compressible flows. Therefore, questions regarding the conditions for the Taylor-Proudman column warrant in-depth investigations, and these constitute the main theme of the present paper.

It is noted that, in most of the published treatises on rotating compressible flows, the steady-state interior flows were treated to be axially (in the z-direction) non-uniform. The results of the present study will provide answers to the question why the z-independent Taylor-Proudman column in the interior did not emerge in the previous accounts on rotating compressible flows. The present study aims to offer physical rationalizations to the key concepts leading to the energy transports between the wall and the boundary-layer flow as well as the interior region.

2. Analysis of flows in a compressible rotating fluid

For $E \ll 1$, the entire flow domain can be divided into five distinct characteristic regions : the interior region [I]; the Ekman horizontal boundary layers of thickness $O(E^{1/2})$ [II & III]; the $E^{1/3}$ inner-Stewartson layer [IV]; and the $E^{1/4}$ outer-Stewartson layer [V] [see, e.g., Sakurai & Matsuda, 1974, Matsuda & Hashimoto, 1976], in which E denotes the Ekman number $E \equiv \mu^* / \rho^*_{00}(r_0 H^*)\Omega^* H^{*2}$.

The solution Φ for a flow variable is written as

$$\Phi = \sum_{n=0}^{\infty} E^{n/2}(\Phi_I^{(n)} + \hat{\Phi}^{(n)}) + \sum_{n=0}^{\infty} E^{n/12}(\overline{\Phi}^{(n)} + \tilde{\Phi}^{(n)}) \quad (1)$$

The well-documented boundary-layer matching technique will be deployed. In the above, Φ stands for the flow variables $\vec{V}$, ρ, p, T. Subscript I denotes the interior region [I], superscript $\hat{}$ the horizontal boundary layer [II & III], superscript $\bar{}$ the $E^{1/3}$ inner-Stewartson layer [IV], and superscript ~ the $E^{1/4}$ outer-Stewartson layer [V].

2.1 interior flow (region I)

Guided by the prior theoretical expositions [e.g., Sakurai & Matsuda, 1974; Bark & Hultgren, 1979], the proper scalings are

$u_I = E^{-1}u,\ v_I = v,\ w_I = E^{-1/2}w,\ T_I = T,\ p_I = p,\ \rho_I = \rho.$

The steady-state leading-order equations are

$$\frac{\partial w_I}{\partial z} = 0, \quad (2a)$$

$$-2\rho_{00} v_I - r\rho_I + \frac{\partial p_I}{\partial r} = 0 \quad , \quad (2b)$$

$$2\rho_{00} u_I = \left(\nabla^2 - \frac{1}{r^2}\right) v_I \quad , \quad (2c)$$

$$\frac{\partial p_I}{\partial z} = 0, \quad (2d)$$

$$-4\alpha^2 r \rho_{00} u_I = \nabla^2 T_I \quad , \quad (2e)$$

$$p_I = \frac{\sigma(\gamma-1)}{4\gamma\alpha^2}(\rho_I + \rho_{00} T_I) \quad . \quad (2f)$$

In the above, $\rho_{00} = \exp\left[\frac{\gamma M^2}{2}\left(\left(\frac{r}{r_0}\right)^2 - 1\right)\right]$, γ the specific ratio, $\sigma \equiv \mu^* C^*_p / k^*$ the Prandtl number and $\alpha^2 = \frac{\sigma(\gamma-1)M^2}{4r_0^2}$.

Clearly, as seen in Eqs.(2a) & (2d), w_I and p_I are the functions of r only.

Placing Eq.(12f) into Eq.(12b) yields

$$v_I(r,z) = \frac{r}{2} T_I(r,z) + \frac{1}{2}\frac{d}{dr}\left(\frac{p_I(r)}{\rho_{00}}\right) \quad . \quad (3)$$

Eq.(3) carries an important physical meaning. The azimuthal velocity in the interior region is determined by two elements : the thermal wind relation, which is represented by the first term on the right-hand-side [RHS] of Eq.(3); and the geostrophic relation, which is shown by the second term of the [RHS] of Eq.(3). The relation in Eq.(3) is termed the thermal geostrophic-wind relation [Matsuda & Hashimoto, 1976], and this characterizes the compressible-fluid flow in the main interior region.

It follows that, by eliminating u_I from Eqs.(2c) and (2e),

$$\nabla^2 T_I + 2\alpha^2 r\left(\nabla^2 - \frac{1}{r^2}\right) v_I = 0 \quad , \quad (4)$$

which was also derived by Sakurai & Matsuda (1974) and Bark & Hultgren (1979).

2.2 Horizontal Ekman boundary layer (region II & III)

Introducing the stretched coordinate

$$\zeta_j = E^{-1/2}(j + (-1)^j z), \quad (j = 0, 1) \quad ,$$

where $j = 0, 1$ denotes respectively the bottom and top horizontal layer, the proper scalings are [see, Sakurai & Matsuda, 1974; Harada, 1979]

$\hat{u} = u,\ \hat{v} = v,\ \hat{w} = E^{-1/2}w,\ \hat{T} = T,\ \hat{\rho} = \rho,\ \hat{p} = E^{-1}p.$

In the above, hat $(\hat{})$ refers to the horizontal boundary layer of strength $O(1)$.

The leading-order steady-state equations are derived as

$$2\rho_{00}\hat{u} = \frac{\partial^2 \hat{v}}{\partial \zeta_j^2}, \quad (5a)$$

$$-4\alpha^2 r \rho_{00}\hat{u} = \frac{\partial^2 \hat{T}}{\partial \zeta_j^2}. \quad (5b)$$

3.3 Vertical boundary layer

As well documented, the vertical Stewartson boundary layer near the sidewall consists of the $E^{1/3}$- inner layer and the $E^{1/4}$-outer layer [e.g., Stewartson, 1957; Bark & Bark, 1976].

(A) $E^{1/3}$-inner layer (region IV)

Introducing the inner-layer coordinate

$$\eta = E^{-1/3}(r_0 - r),$$

the proper scalings are [Sakurai & Matsuda, 1974; Bark & Bark, 1976]

$\bar{u} = E^{-1/3}u,\ \bar{v} = v,\ \bar{w} = w,\ \bar{T} = T,\ \bar{\rho} = \rho,\ \bar{p} = E^{-1/3}p.$

In the above, overbar denotes the inner-layer of strength $O(1)$.

The leading-order steady-state equations are obtained :

$$2\rho_{00}\bar{u} = \frac{\partial^2 \bar{v}}{\partial \eta^2}, \quad (6a)$$

$$-4\alpha^2 r \rho_{00}\bar{u} = \frac{\partial^2 \bar{T}}{\partial \eta^2} \quad . \quad (6b)$$

(B) $E^{1/4}$-outer layer (region V)

Introducing the outer-layer coordinate

$$\xi = E^{-1/4}(r_0 - r),$$

the appropriate scalings are found [Matsuda & Hashimoto, 1978; Matsuda & Takeda, 1978; Park & Hyun, 1997, 1998]

$\tilde{u} = E^{-1/2}u,\ \tilde{v} = v,\ \tilde{w} = E^{-1/4}w,\ \tilde{T} = T,\ \tilde{\rho} = \rho,\ \tilde{p} = E^{-1/4}p.$

In the above, tilde refers to the outer layer of strength $O(1)$.

the leading-order steady-state equations are secured :

$$2\rho_{00}\tilde{u} = \frac{\partial^2 \tilde{v}}{\partial \xi^2}, \quad (7a)$$

$$-4\alpha^2 r\rho_{00}\tilde{u}=\frac{\partial^2\tilde{T}}{\partial\xi^2}. \tag{7b}$$

3. Condition for the Taylor-Proudman column

In the ensuing analysis, the energy content e is defined as

$$e\equiv T+2\alpha^2 rv\ , \tag{8}$$

in which r denotes the radial coordinate, T the temperature, v the azimuthal velocity in the rotating frame, and $\alpha^2=\sigma(\gamma-1)M^2/4r_0^2$. The concept of the energy content is derived from the energy transport analysis of Park & Hyun(2002). It has an important physical meaning in that the normal gradient of energy content at the horizontal wall represents the energy transport rate across the horizontal layer. The energy content is shown to be a physical property which is very useful in the study of compressible rotating flows.

Utilizing the afore-defined energy content, $e_I=T_I+2\alpha^2 rv_I$, in the interior region, Eq.(3) leads to the relations

$$T_I=\left[e_I-\alpha^2 r\frac{d}{dr}\left(\frac{p_I}{\rho_{00}}\right)\right]/(1+\alpha^2 r^2)\ , \tag{9a}$$

$$v_I=\left[\frac{r}{2}e_I+\frac{1}{2}\frac{d}{dr}\left(\frac{p_I}{\rho_{00}}\right)\right]/(1+\alpha^2 r^2)\ . \tag{9b}$$

By substituting Eq.(9a, b) into Eq.(4), the equation governing e_I is found :

$$\frac{\partial}{\partial r}\left[r(1+\alpha^2 r^2)\frac{\partial}{\partial r}\left(\frac{e_I}{1+\alpha^2 r^2}\right)\right]+r\frac{\partial^2 e_I}{\partial z^2}=\frac{d}{dr}\left[\frac{2\alpha^2 r}{1+\alpha^2 r^2}\frac{d}{dr}\left(\frac{p_I}{\rho_{00}}\right)\right]. \tag{10}$$

To obtain the associated boundary conditions for eq.(10), the boundary-layer matching conditions are derived :

at $z=0$, $e_I(r,z=0)+\hat{e}(r,\zeta_0\to 0)=e_{BW}$, (11a)

at $z=1$, $e_I(r,z=1)+\hat{e}(r,\zeta_1\to 0)=e_{TW}$, (11b)

at $r=r_0$, $e_I(r_0,z)+\bar{e}(\eta\to 0,z)+\tilde{e}(\xi\to 0,z)=e_{VW}$ (11c)

and $\hat{e}(r,\zeta_0\to\infty)=\hat{e}(r,\zeta_1\to\infty)=\bar{e}(\eta\to\infty,z)=\tilde{e}(\xi\to\infty,z)=0$, (11d)

in which BW, TW and VW denote, respectively, the horizontal bottom wall, the horizontal top wall and the vertical wall. Eq.(11d) stems from the behavior of the boundary layer variables.

By algebraic manipulations of Eqs.(5a,b), the equation for the energy content $\hat{e}[\equiv\hat{T}+2\alpha^2 r\hat{v}]$ in the horizontal boundary layer is arrived at :

$$\frac{\partial^2\hat{e}}{\partial\zeta_j^2}=0. \tag{12}$$

Together with the boundary condition (11d), i.e., as $\zeta_j\to\infty$ ($j=0, 1$), $\hat{e}\to 0$, it is clear that, for arbitrary ζ_j,

$$\frac{\partial\hat{e}}{\partial\zeta_j}=0, \tag{13a}$$

and $\hat{e}=0$. (13b)

In a similar manner, the energy contents $\bar{e}[\equiv\bar{T}+2\alpha^2\bar{v}]$ and $\tilde{e}[\equiv\tilde{T}+2\alpha^2\tilde{v}]$ in the vertical boundary layers are shown to satisfy [see Eqs.(6) and (7)]

$$\frac{\partial^2\bar{e}}{\partial\eta^2}=0, \tag{14}$$

$$\frac{\partial^2\tilde{e}}{\partial\xi^2}=0. \tag{15}$$

In view of the boundary conditions (11d), i.e., as $\eta\to\infty$, $\bar{e}\to 0$, and as $\xi\to\infty$, $\tilde{e}\to 0$, one obtains

$$\bar{e}=0, \tag{16}$$

$$\tilde{e}=0. \tag{17}$$

The preceding analyses indicate that the boundary-layer energy contents are identically zero, i.e., $\hat{e}(\zeta_j)=\bar{e}(\eta)=\tilde{e}(\xi)=0$. Therefore, the boundary condition for equation (10), which governs the energy content $e_I[\equiv T_I+2\alpha^2 rv_I]$ in the main interior region, can be stated from Eqs.(11a-c) as, at all the surface walls of the container, $e_I=T_W+2\alpha^2 rV_W$. (18)

In the above, T_W and V_W denote respectively the perturbed temperature and azimuthal velocity boundary conditions imposed on the container wall.

Now, specialization is made to the Taylor-Proudman column, which demands uniformity in the axial direction in the interior region, i.e., $\frac{\partial e_I}{\partial z}=0$. (19)

Thus, in order to sustain the Taylor-Proudman column in the interior region, from Eqs.(13a) and (19), the boundary condition for the energy content e at the horizontal wall should satisfy

$$\left(\frac{\partial e}{\partial z}\right)_{\text{at horizontal wall}}=\left(\frac{\partial e_I}{\partial z}\right)_{\text{at horizontal wall}}+E^{1/2}\left(\frac{\partial\hat{e}}{\partial\zeta_j}\right)_{\text{at horizontal wall}}=0. \tag{20}$$

The result of Eq.(20) is identical to the condition for the Taylor-Proudman column in P&H. This exercise reconfirms the effectiveness of the concept of the energy content e.

4. Admissible Taylor-Proudman column flows

A feasible solution under condition (20) or (21) is considered. From Eq.(10), one has

$$\frac{de_I}{dr}-\frac{2\alpha^2 r}{1+\alpha^2 r^2}e_I=\frac{2\alpha^2}{1+\alpha^2 r^2}\frac{d}{dr}\left(\frac{p_I}{\rho_{00}}\right). \tag{21}$$

In the interior region , the pressure satisfies the following relation [see, Hashimoto, 1977; Bark & Hultgren, 1979]:

$$\frac{d}{dr}\left(\frac{p_I}{\rho_{00}}\right)=V_{TW}+V_{BW}-\frac{r}{2}(T_{TW}+T_{BW}). \tag{22}$$

And, from the boundary condition (11a), $e_I(r)$ has to satisfy

$$e_I(r)=T_{BW}(r)+2\alpha^2 rV_{BW}(r)=T_{TW}(r)+2\alpha^2 rV_{TW}(r)\ . \tag{23}$$

Based on Eqs.(22) & (23), one obtains

$$\frac{d}{dr}\left(\frac{p_I}{\rho_{00}}\right)=(1+\alpha^2 r^2)(V_{TW}+V_{BW})-re_I\ . \tag{24}$$

Combining the above developments, an equation for the energy content e_I in the interior is secured from Eq.(21):

$$\frac{de_I}{dr}=2\alpha^2(V_{TW}+V_{BW})=4\alpha^2 V_S\ . \tag{25}$$

The solution to Eq.(25) is

$$e_I=4\alpha^2\int V_S dr\ . \tag{26}$$

The associate conditions are, from Eqs.(23) and (26),

$$T_A=-2\alpha^2 rV_A\ , \tag{27a}$$

$$T_S=4\alpha^2\int V_S dr-2\alpha^2 rV_S\ , \tag{27b}$$

and, from Eqs.(11b) and (26),

at $r=r_0$, $e_{VW}=e_I\left[\equiv 4\alpha^2\int V_S dr\right]$, (27c)

in which subscripts A and S denote, respectively, the asymmetric and symmetric components.

In summary, in order to maintain the Tayor-Proudman column in the interior, the boundary conditions imposed at the wall should satisfy Eqs.(27a-c). Consider an arbitrary velocity boundary condition at the wall. For this velocity condition, there exist always a corresponding temperature condition which would satisfy Eqs.(27a-c). A multitude of different theoretical solutions can be obtained. Provided that the velocity and temperature boundary conditions satisfy Eqs.(27a-c), the solutions for the interior flow can be found by substituting Eqs.(24) & (26) into Eqs.(9a, b), i.e.,

$$T_I=\frac{T_{TW}+T_{BW}}{2}[\equiv T_S], \quad (28a)$$

$$v_I=\frac{V_{TW}+V_{BW}}{2}[\equiv V_S]. \quad (28b)$$

The complete global solution, which includes the solutions in the horizontal and vertical boundary layers, can now be secured by the straightforward methods that have been discussed in sufficient detail in the literature [e.g., Sakurai & Matsuda, 1974; Bark & Bark, 1976; Park & Hyun, 1997, 1998], and they will not be included in the present paper. In passing, it is worth mentioning that the theoretical methodologies of the present paper lessen the mathematical difficulties which arise in the coupling process of the boundary-layer and interior flows as in the previous papers. In fact, in the present effort, it was shown that the interior solution may be obtained without having to work through the boundary-layer flow analyses.

5. Conclusions

The conditions for the Taylor-Proudman column in the main interior have been studied by the matched asymptotic analysis.

The energy content e was re-formulated, and its effectiveness in characterizing the energy transport mechanism was delineated. It has an important physical meaning in that the normal gradient of energy content at the horizontal wall represents the energy transport rate across the horizontal layer. The energy content is a kind of physical property which turns out to be very useful in the study of compressible rotating flows.

In general compressible rotating flows, the values of boundary-layer energy content, i.e., $\hat{e}$ in the horizontal layer, $\bar{e}$ in the $E^{1/3}$ vertical layer and $\tilde{e}$ in the $E^{1/4}$ vertical layer, are identically zero, which means that the interior energy content e_I should satisfy, in itself, the natural boundary condition at the container wall, i.e., $e_I=T_W+2\alpha^2 r V_W$ at all the surface walls of the container. The above property of energy content allows an easy analysis of the interior flow, since detailed boundary-layer flow analyses are not needed.

For the steady-state Taylor-Proudman column to be maintained, the condition $\left(\frac{\partial e}{\partial z}\right)_{at\ horizontal\ wall}=0$ has to be met. This implies that the net energy transfer rate between the horizontal wall and the interior fluid vanishes. The outcome of the present matched asymptotic analysis was shown to be consistent with the findings of Park & Hyun(2002) which utilized the energy balancing analysis.

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