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Energy transport analysis for the Taylor-Proudman column in a rapidly-rotating compressible fluid

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압축성 회전 유동에서의 Taylor-Proudman 기둥의 에너지 전달에 관한 해석

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Abstract

A theoretical study is made of the steady flow of a compressible fluid in a rapidly rotating finite cylinder. Flow is generated by imposing mechanical and/or thermal disturbances at the rotating endwall disks. Both the Ekman and Rossby numbers are small. A detailed consideration is given to the energy budget for a control volume in the Ekman boundary layer. A combination of physical variables, which is termed the energy contents, consisting of temperature and modified angular momentum, emerges to be relevant. The distinguishing features of a compressible fluid, in contrast to those of an incompressible fluid, are noted. For the Taylor-Proudman column to be sustained, in the interior, it is shown that the net energy transport between the solid disk wall and the interior fluid should vanish. Physical rationalizations are facilitated by resorting to the concept of the afore-stated energy content.

1. Introduction

Flow of a fluid in a finite, closed cylinder (radius $R^*_0[\equiv r_0 H^*]$, height H^*), which rotates steadily about its longitudinal axis, has posed a classical problem. For most applications, the system Ekman number $E[\equiv \mu^*/\rho^*_{00}\Omega^* H^{*2}$, where μ^* denotes the viscosity of fluid, ρ^*_{00} the reference density of fluid at the cylinder wall, Ω^* the representative rotation rate of the cylinder] is very small. When all the components of the solid walls of the container rotate in unison, the fluid is in rigid-body rotation with rotation rate Ω^*, and, therefore, no internal flows in the rotating frame exist. However, if there are discrepancies $\triangle\Omega^*$ between the rotation rates of the components of the container walls, flows are generated in the fluid. The departure from the rigid-body rotation is gauged by the Rossby number $\varepsilon[\equiv \triangle\Omega^*/\Omega^*]$. Attention is focused to the practically interesting cases of a rapidly-rotating cylinder, $E\ll 1$, $\varepsilon\ll 1$, and studies have described salient characteristics of flows in the boundary layers and in the interior region. In particular, knowledge of the Ekman layer and the interior flow constitutes an essential building block in the understanding of rotating fluid machinery operations and geophysical fluid dynamics [see, e.g., Greenspan, 1968; Roberts & Soward, 1978].

One celebrated flow pattern pertinent to $E\ll 1$, $\varepsilon\ll 1$ is the Taylor-Proudman column. This is characterized by the uniformity of velocities in the axial direction in the inviscid interior region, which is far away from the solid walls. However, it should be pointed out that the Taylor-Proudman column is attainable only when the fluid is incompressible and of constant density. In this case, two forces in the radial direction are in balance in the inviscid interior, i.e., the Coriolis force and the pressure gradient, which leads to the geostrophy [see, e.g., Greenspan, 1968; Spohn, Mory & Hopfinger, 1998].

The purpose of the present paper is to undertake a theoretical examination of the principal mechanisms of flow and energy transports of a compressible fluid in a rapidly-rotating finite cylinder. Based on the analytical expositions, the precise conditions for the Taylor-Proudman column in the steady state will be delineated. The theoretical endeavors will illuminate the significance of a particular combination of physical variables, which turns out to be effective in rationalizing some of the general features of rotating compressible flows.

2. The mathematical model

At the initial state, a compressible fluid is in a cylinder, which rotates about the longitudinal z^*-axis with rotation rate Ω^*. The cylindrical coordinates (r^*, θ, z^*), together with the velocity components (u^*, v^*, w^*) viewed from the rotating frame,

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are adopted. For convenience, the z^*-axis is aligned in the vertical direction. The fluid is in isothermal equilibrium with the container walls at temperature T^*_{00}. Assuming a perfect gas, the density of fluid in the basic state is [e.g., Sakurai & Matsuda, 1974; Bark & Bark, 1976] :

$$\rho^*_{00}(r^*)=\rho^*_{00}(r_0 H^*)\exp\left[\frac{\gamma M^2}{2}\left(\left(\frac{r}{r_0}\right)^2-1\right)\right], \quad (1)$$

and the pressure field is given as

$$p^*_{00}(r^*)=\rho^*_{00}(r^*)RT^*_{00}. \quad (2)$$

In the above, superscript * denotes dimensional quantities, and subscript 00 the initial-state rigid-body rotation, $r=r^*/H^*$, $M[\equiv\Omega^* r_0 H^*/(\gamma R T^*_{00})^{1/2}]$ the Mach number, γ the ratio of specific heats, $r_0\equiv R^*_0/H^*$, and R the gas constant.

Departures from the above initial rigid-body rotation are created by applying small thermal and/or mechanical perturbations to the walls of the cylinder. The relative strength of perturbation is measured by the Rossby number $\varepsilon\equiv T^{*p}/T^*_{00}$ (or $U^{*p}/(\Omega^* H^*)$) , where T^{*p} (or U^{*p}) indicates the magnitude of thermal (or mechanical) perturbation at the walls. It follows that, for $\varepsilon\ll 1$, as viewed from the frame rotating at Ω^*, the dependent variables are $O(\varepsilon)$. Neglecting the $O(\varepsilon^2)$ and higher-order terms, the dimensional linearized governing equations are [e.g., Morberg et al., 1984] :

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*\rho^*_{00}u^*)+\rho^*_{00}\frac{\partial w^*}{\partial z^*}=0, \quad (3)$$

$$-2\rho^*_{00}\Omega^* v^*-\Omega^{*2}r^*\rho^*=-\frac{\partial p^*}{\partial r^*}$$

$$+\mu^*\left[\left(\nabla^2-\frac{1}{r^{*2}}\right)u^*+\left(\frac{1}{3}+\beta\right)\frac{\partial}{\partial r^*}(\nabla\cdot\vec{V}^*)\right], \quad (4)$$

$$2\rho^*_{00}\Omega^* u^*=\mu^*\left(\nabla^2-\frac{1}{r^{*2}}\right)v^*, \quad (5)$$

$$\frac{\partial p^*}{\partial z^*}=\mu^*\left[\nabla^2 w^*+\left(\frac{1}{3}+\beta\right)\frac{\partial}{\partial z^*}(\nabla\cdot\vec{V}^*)\right], \quad (6)$$

$$-\Omega^{*2}r^*\rho^*_{00}u^*=\kappa^*\nabla^2 T^*, \quad (7)$$

$$p^*=R(\rho^*_{00}T^*+\rho^* T^*_{00}). \quad (8)$$

in which β denotes the thermal expansion coefficient, μ^* the coefficient of viscosity, k^* the coefficient of thermal conductivity.

The boundary conditions in dimensional form are :
at the bottom endwall disk $z^*=0$,

$$u^*=w^*=0,\quad v^*=V^*_{BW}(r^*),\quad T^*=T^*_{BW}(r^*), \quad (9a)$$

at the top endwall disk $z^*=H^*$,

$$u^*=w^*=0,\quad v^*=V^*_{TW}(r^*),\quad T^*=T^*_{TW}(r^*), \quad (9b)$$

and at the cylindrical wall $r^*=r_0 H^*$,

$$u^*=w^*=0,\quad v^*=V^*_{VW}(z^*),\quad T^*=T^*_{VW}(z^*), \quad (9c)$$

in which BW, TW, VW denote, respectively, the bottom horizontal disk, the top horizontal disk and the cylindrical vertical sidewall.

It is advantageous to split the horizontal boundary conditions into symmetric and anti-symmetric parts, i.e.,

at $z^*=0$, $u^*=w^*=0$, $v^*=V^*_S-V^*_A$,
$T^*=T^*_S-T^*_A$, (10a)

at $z^*=H^*$, $u^*=w^*=0$, $v^*=V^*_S+V^*_A$,
$T^*=T^*_S+T^*_A$, (10b)

at $r^*=r_0 H^*$, $u^*=w^*=0$, $v^*=V^*_{VW}$, $T^*=T^*_{VW}$ (10c)

in which $V^*_S=(V^*_{TW}+V^*_{BW})/2$,
$V^*_A=(V^*_{TW}-V^*_{BW})/2$,
$T^*_S=(T^*_{TW}+T^*_{BW})/2$,
$T^*_A=(T^*_{TW}-T^*_{BW})/2$,

and subscripts A and S denote respectively the anti-symmetric and symmetric parts.

3. Energy budget in the horizontal boundary layer

It is meaningful to consider the balance of the steady-state energy transport in the horizontal boundary layer on the endwall disk when a mechanical and/or thermal perturbation is given. For this purpose, the control volume consists of the annular zone in the horizontal boundary layer with inner and outer radii (r^*, r^*+dr^*). The transports of energy into and out of the control volume are accounted for, and they consist of three elements : (1) mechanical work done by the viscous friction at the control surface, ΣW^*_f; (2) heat transfer due to the temperature difference at the boundary of control volume, ΣQ^*; and (3) energy flux by fluid motion across the control surface, ΣW^*_p. Item(3) is closely related to the work done by the basic radial pressure gradient when the fluid moves in the radial direction. This arises when the fluid in the horizontal boundary layer undergoes radial (horizontal) motions in the environment of the initially-given radial pressure gradient described in Eqs.(1) & (2). The role of ΣW^*_p is crucial in the dynamics of a compressible fluid. The radially-moving fluid does the pressure-gradient work, ΣW^*_p, and this is associated with the compression (or expansion), which leads to the heating (or cooling). The resultant effect is akin to that of a heat source (or sink) placed inside the boundary layer. It is important to recognize that, in the case of a compressible fluid, temperature changes occur in the interior even when there are no temperature changes imposed at the boundary walls. Consequently, the velocity and temperature fields in the interior are generally z^*-dependent. This is a significant feature which distinguishes a compressible-fluid motion from that of an incompressible fluid.

In the ensuing discussion, the thickness of horizontal boundary layer is $\delta^*_H\sim O(E^{1/2}H^*)$ [see, Sakurai & Matsuda, 1974; Harada, 1979]. Under the assumption of $\varepsilon\ll 1$, $dr^*\ll 1$, the higher order terms will be neglected.

The rate of mechanical work, ΣW^*_f, transferred from the exterior to the interior of the control volume, is

$$\Sigma W^*_f=\mu^*\oint_S(\vec{n}^*\cdot\nabla\vec{V}^*_{inertial})\cdot\vec{V}^*_{inertial}dS^*.$$

Above integration can be approximated as

$$\Sigma W_f^* \cong (W_f^*)_{z^*=\delta_H^*} - (W_f^*)_{z^*=0} + O(\varepsilon^2, \varepsilon E^{1/2})$$
$$\cong \mu^* \varepsilon \left[\left(\frac{\partial v^{*p}_{rotating}}{\partial z^*} \right)_{z^*=\delta_H^*} - \left(\frac{\partial v^{*p}_{rotating}}{\partial z^*} \right)_{z^*=0} \right] (\Omega^* r^*) A^* \qquad (11)$$

In the above, $A^* = 2\pi r^* d r^*$.

The rate of heat transfer ΣQ^*, transported from the exterior to the interior of the control volume, is

$$\Sigma Q^* = \varepsilon k^* \oint_{S^*} \vec{n}^* \cdot \nabla T^* dS^*.$$

The above integration can be approximated as

$$\Sigma Q^* \cong (Q^*)_{z^*=\delta_H^*} - (Q^*)_{z^*=0} + O(\varepsilon E^{1/2})$$
$$\cong \varepsilon k^* \left[\left(\frac{\partial T^{*p}}{\partial z^*} \right)_{z^*=\delta_H^*} - \left(\frac{\partial T^{*p}}{\partial z^*} \right)_{z^*=0} \right] A^*. \qquad (12)$$

The energy flux Σw^*_P is carried by the velocity normal to the control surface. The energy per unit mass is

$$h^* = \frac{\vec{V}^*_{inertial} \cdot \vec{V}^*_{inertial}}{2} + C_p T^*$$
$$\cong \frac{(\Omega^* r^*)^2}{2} + C_p T^*_{00}.$$

Thus, the energy flux to the exterior from the interior of the control volume is

$$\Sigma W_p^* = \oint_{S^*} (\vec{V}^*_{inertial} \cdot \vec{n}^*) h^* dS^*. \qquad (13)$$

The above integration on the control surface, after some mathematical manipulations, becomes

$$\Sigma W_p^* \cong \varepsilon \frac{\mu^*}{2} \left[\left(\frac{\partial v^{*p}_{rotating}}{\partial z^*} \right)_{z^*=\delta_H^*} - \left(\frac{\partial v^{*p}_{rotating}}{\partial z^*} \right)_{z^*=0} \right] (\Omega^* r^*) A^* \qquad (14)$$

In summary, under $\varepsilon \ll 1$, $E \ll 1$, in the steady horizontal boundary layer, the net total energy transport ΣE^* to the control volume at an arbitrary radial position r^* should be zero :

$$\Sigma E^* = \Sigma W_f^* + \Sigma Q^* - \Sigma W_p^* = 0 \ , \qquad (15)$$

which gives

$$\varepsilon \frac{\partial}{\partial z^*} \left(\frac{1}{2} \Omega^* r^* \mu^* v^{*p}_{rotating} + k^* T^{*p} \right)_{at\ z^*=\delta_H^*}$$
$$= \varepsilon \frac{\partial}{\partial z^*} \left(\frac{1}{2} \Omega^* r^* \mu^* v^{*p}_{rotating} + k^* T^{*p} \right)_{at\ z^*=0}. \qquad (16)$$

A physical rationalization for Eq.(16) is in order. For a steady flow in the horizontal boundary layer of a compressible rotating fluid, the vertical gradients of the quantity $\varepsilon \left(\frac{1}{2} \Omega^* r^* \mu^* v^{*p}_{rotating} + k^* T^{*p} \right)$ at the horizontal wall and at the edge of the boundary layer should be the same. It is recalled that the normal gradient of the quantity $\varepsilon \left(\frac{1}{2} \Omega^* r^* \mu^* v^{*p}_{rotating} + k^* T^{*p} \right)$ indicates the rate of energy transport across the control surface. Consequently, Eq.(16) states that the steady Ekman boundary layer of a compressible fluid delivers a part of energy that it receives from the wall (from $r^*=0$ to $r^*=r^*$) to the interior across the edge of the boundary layer (from $r^*=0$ to $r^*=r^*$).

In essence, the total energy transport from the horizontal wall to the interior is expressed as the normal gradient of the physical variables grouped in the parenthesis of Eq.(26), i.e.,

$$\varepsilon \left(\frac{1}{2} \Omega^* r^* \mu^* v^{*p}_{rotating} + k^* T^{*p} \right). \qquad (17)$$

By properly nondimensinalizing the quantities in Eq.(17) as $r = r^*/H^*$, $v = v^{*p}_{rotating}/\Omega^* H^*$, $T = T^{*p}/T^*_{00}$, a nondimensional quantity, which will, for convenience, be termed the energy content, is obtained :

$$e[\equiv \frac{\varepsilon\left(\frac{1}{2} \Omega^* r^* \mu^* v^{*p}_{rotating} + k^* T^{*p} \right)}{\varepsilon k^* T^*_{00}}] \equiv T + 2\alpha^2 rv. \qquad (18)$$

In the above, $\alpha^2 \equiv \sigma(\gamma-1) M^2/4 r_0^2$, and v represents the azimuthal velocity in the rotating frame. $\sigma \equiv \mu^* C^*_p / k^*$ the Prandtl number, and C^*_p denotes the specific heat at constant pressure. For simplicity, the superscript "p" and subscript "rotating" are omitted in nondimensional variables.

The concept of the energy content e will be of considerable usefulness. Many of the previously-established flow phenomena can be explained in a straightforward manner by using e. The introduction of e will aid in rationalizing more fundamental issues of general rotating compressible-fluid dynamics.

4. Conditions for the Taylor-Proudman column flow

If the Taylor-Proudman column flow prevails in the interior, one has

$$\left(\frac{\partial v^*_{rotating}}{\partial z^*} \right)_{at\ z^*=\delta_H^*} = 0\ . \qquad (19a)$$

In the interior, the viscous term is smaller than the other terms, i.e., $(viscous\ term)/(the\ other\ term) \sim O(E)$. Neglecting the viscous term in Eq.(6), the pressure in the interior is only a function of r^*, i.e., $\partial p^*/\partial z^* = 0$. Utilizing $p^*(r^*)$ in Eq.(8), Eq.(4) gives the relation in the interior that $\frac{\partial v^*_{rotating}}{\partial z^*} = \frac{r^*}{2T^*_{00}} \frac{\partial T^*}{\partial z^*}$ [see Sakurai & Matsuda, 1974; Bark & Hultgren, 1979]. Thus, if the interior flow is assumed to be the Taylor-Proudman column, the temperature condition below should satisfy, from Eq.(19a):

$$\left(\frac{\partial T^*}{\partial z^*} \right)_{at\ z^*=\delta_H^*} = 0. \qquad (19b)$$

Substituting Eqs.(29a) and (29b) into Eq.(26) leads to

$$\varepsilon \frac{\partial}{\partial z^*} \left(\frac{1}{2} \Omega^* r^* \mu^* v^*_{rotating} + k^* T^* \right)_{at\ horizontal\ wall} = 0. \qquad (20)$$

The foregoing developments point to the conditions for the Taylor-Proudman column flow of a compressible fluid. It also stresses the significance of introducing the concepts of energy transports in the context of the Taylor-Proudman column. This turns up in the definition of the afore-stated energy content $e[\equiv T + 2\alpha^2 rv]$, which consists of temperature and angular *momentum* modified by the compressibility effect. In view of the above derivations, the condition to maintain the steady Taylor-Proudman column in the interior can be readily found from Eqs.(19a) and (19b) : *the net total energy transports should be* confined within the horizontal boundary layer, i.e., from Eqs.(18)

and (20)

$$\left(\frac{\partial e}{\partial z}\right)_{at\ horizontal\ wall}=0. \qquad (21)$$

Clearly, in the above, $z[\equiv z^*/H^*]$ denotes the outward coordinate normal to the endwall disk.

Eq.(21) will be stated as the conditions for the Taylor-Proudman column in a compressible rotating fluid.

5. Conclusion

Characteristics of energy transfer in a rapidly-rotating compressible fluid are depicted. There are two types of energy transfer mechanisms between the rotating disk wall and the fluid : one is a energy transfer normal to the disk wall, which is characterized by the gradient of energy content $e[\equiv T+2\alpha^2 rv]$; the other is a horizontal convection process, which is very similar to the typical Ekman layer flow in an incompressible fluid. However, in rapidly-rotating compressible flows, fluid motions in the radial direction give rise to the generation (removal) of heat owing to the compression (expansion) work. These interactions between the temperature and velocity fields render the problem to be more complex.

In the horizontal boundary layer, when the Taylor-Proudman column prevails in the interior flow, the total energy in the steady state is balanced as follows : a half of the mechanical energy input from the rotating horizontal disk is consumed to overcome the basic radial pressure gradient and to sustain the Ekman boundary layer flow in the radial direction. The rest of the mechanical energy input is converted into the thermal energy in the Ekman layer and is withdrawn to the disk wall.

By undertaking a detailed analysis of energy transports in the boundary layer, the conditions to maintain the steady Taylor-Proudman column in the interior are derived: the normal gradient of energy content e should be identically zero at the horizontal wall disk, i.e., $(\vec{n}\cdot\nabla e)_{at\ the\ horizontal\ wall}=0$. In this case, the energy content e should satisfy the three characteristic properties : (1) the values of the energy content e should be the same at both the top and bottom horizontal disks at $r=r$; (2) the value of e in the interior at $r=r$ is identical to the value e at the horizontal disk,; (3) the value of e at the vertical wall should be the same as the value of e at the horizontal disk at $r=r_0$. Under the condition of the Taylor-Proudman column, all the energy transports are accomplished through the boundary layer.

The concept of the energy variable e is very useful in rationalizing these physical processes.

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