Proceedings of Joint Conference of Korean Data And Information Science Society and Korean Data Analysis Society, June 21-22, 2002, pp. 17-22

On the Steady State Availability of Age-Dependent Minimal Repair Model

Ji Hwan Cha[1], *Jae Joo Kim*[2]

Abstract

Availability is an important characteristic of a repairable component. Iyer(1992) obtained the 'limiting efficiency'(not the `steady state availability') of the age-dependent minimal repair model which was first considered by Block et al.(1985). However the existence of the steady state availability of the model has not been reported. In this note, the existence of the steady state availability of the model is shown and a brief remark on the importance of the property is given.

Keywords : Steady State Availability, Limiting Efficiency, Repairable Component, Minimal Repair, Perfect Repair, Imperfect Repair

1. Introduction

Consider a component which can be in one of two states, namely `up' and `down'. By `up' we mean the component is still functioning and by `down' we mean the component is not functioning; in the latter case the component is being repaired or replaced, depending on whether the component is repairable or not. Let the state of the component be given by the binary variable

$$X(t) = \begin{cases} 1 & \text{if } \textit{the component is up at time } t \\ 0 & \textit{otherwise.} \end{cases} \tag{1}$$

An important characteristic of a repairable component is availability. The availability at time t is defined by

$$A(t) = P(X(t) = 1), \tag{2}$$

which is the probability that the component is functioning at time t. Because the study of

[1]Department of Statistical Information, Catholic University of Daegu, Kyeoungbook 712-702, Korea
E-mail : jhcha@cuth.cataegu.ac.kr
[2]Department of Statistics, Seoul National University, Seoul 151-742, Korea
E-mail : jjkim@stats.snu.ac.kr

$A(t)$ is too hard except for a few simple cases, other measures have been proposed, and more attention is being paid to the limiting behavior of this quantity, i.e., engineers are more interested in the extent to which the component will be available after it has been run for a long time. The steady state availability (or limiting availability) of the component is, when the limit exists, defined by

$$A = \lim_{t \to \infty} A(t), \tag{3}$$

which is a significant measure of performance of a repairable component. Some other kinds of availability which are useful in practical applications can be found in Birolini(1985, 1994) and Hoyland and Rausand(1994).

Another measure of performance associated with a repairable component is efficiency defined by

$$Eff_t = \frac{E(U(t))}{t}, \tag{4}$$

where $U(t)$ is the total amount of functioning time during $(0, t]$, that is, $U(t) = \int_0^t X(u)du$. This measure can be interpreted as the expected proportion of time the component is functioning during $(0, t]$. The limiting efficiency is defined by

$$Eff_\infty = \lim_{t \to \infty} Eff_t = \lim_{t \to \infty} \frac{E(U(t))}{t}, \tag{5}$$

which was first considered by Barlow and Hunter(1960).

Note that if $A = \lim_{t \to \infty} A(t)$ exists, then Eff_∞ also exists and the relationship $\lim_{t \to \infty} A(t) = Eff_\infty$ holds. However, the $\lim_{t \to \infty} A(t)$ does not always exist even if Eff_∞ exists (See Barlow and Proschan, 1975, ch. 7).

Brown and Proschan(1983) considered the imperfect repair model where a component is renewed with probability p and is minimally repaired with probability $1-p$, on each failure. This model was extended to the age-dependent minimal repair model by Block et al.(1985), where p depends on the component age t. Iyer(1992) obtained the limiting efficiency of the age-dependent minimal repair model based on a Renewal Reward Process, but, as Mi(1998, p211) pointed out, Iyer(1992) inappropriately called the quantity the steady state availability.

In this note, the existence of the steady state availability of the age-dependent minimal repair model is shown and a brief remark on the importance of the property is given.

2. Main Result

An availability model for the age-dependent minimal repair model will be developed under the assumption of non-negligible repair times. A piece of component with failure rate function $r(t)$ is put into operation at time 0. Each time it fails, a maintenance action is taken which, with probability $p(t)$, is a complete repair or, with probability $q(t) = 1 - p(t)$, is a minimal repair, where t is the age at failure of the component under maintenance. Let V_i be the time of the completion of i^{th} renewal, $i = 1, 2, \cdots$. Assume that after each renewal, the component state is as good as new state. Define N_i as the number of failures in the i^{th} renewal period, $(V_{i-1}, V_i]$, $i = 1, 2, \cdots$, where $V_0 \equiv 0$. Let T_{ij} be the lifetime of the component which has been renewed $i - 1$ times and has been minimally repaired $j - 1$ times after the time of the $i - 1^{th}$ renewal, $i = 1, 2, \cdots$, $j = 1, 2, \cdots, N_i$. Let the conditional distribution function of T_{ij} given $N_i = r$ be $F_{(r)j}(x)$, and $E(T_{ij} | N_i = r) \equiv \mu_{(r)j}$, $i = 1, 2, \cdots$, $j = 1, 2, \cdots, r$, $r = 1, 2, \cdots$. Let us define the corresponding repair times R_{ij} and assume that the distribution of R_{ij} is, $G_1(y)$ with mean ν_1 for $i = 1, 2, \cdots$, $j = 1, 2, \cdots, N_i - 1$, and $G_2(y)$ with mean ν_2 for $i = 1, 2, \cdots$, $j = N_i$. This means that the times for minimal repairs are identically distributed according to $G_1(y)$ and the distribution of times for renewals is $G_2(y)$. Furthermore we assume that R_{ij}'s are finite with probability 1.

Also define $Z_{ij} \equiv \sum_{m=1}^{j} (T_{i,m} + R_{i,m})$, $j = 1, 2, \cdots, N_i - 1$, and $Z_i \equiv \sum_{m=1}^{N_i} (T_{i,j} + R_{i,m})$, which is the length of the i^{th} renewal period. Let the conditional distribution function of $Z_{i,j}$ given $N_i = r$ be $F_{(r)Z_{i,j}}(t)$ $i = 1, 2, \cdots$, $j = 1, 2, \cdots, r - 1$, $r = 2, \cdots$, and the distribution of Z_i be $H(t)$, $i = 1, 2, \cdots$. Assume that the Z_i's are mutually independent for $i = 1, 2, \cdots$. Obviously, V_n is given by $V_n = \sum_{i=1}^{n} Z_i$, $n = 1, 2, \cdots$. Define $H^{(n)}(t)$ as n-fold convolution of $H(t)$ and $M_H(t)$ as $\sum_{n=1}^{\infty} H^{(n)}(t)$.

Observe that for the cases of $N_i = 1$, the corresponding renewal period consists of $T_{i,1}$ and $R_{i,1}$, and their distribution functions are $F_{(1)1}(x)$ and $G_2(y)$ respectively.

The existence of the steady state availability of the age-dependent minimal repair model is shown in the following theorem.

Theorem 2.1. Assume that $\int_0^{\infty} p(u) r(u) du = \infty$ then the steady state availability of the

Age-Dependent Minimal Repair Model exists and is given by

$$\begin{aligned} A &= \lim_{t\to\infty} A(t) \\ &= \frac{\mu(p)}{\mu(p) + \mu_{N-1}\nu_1 + \nu_2}, \end{aligned} \tag{6}$$

where

$$\mu(p) = \int_0^{\infty} \exp\left(-\int_0^t p(u)r(u)du\right)dt$$

and

$$\mu_{N-1} = \int_0^{\infty} q(t)r(t)\exp\left(-\int_0^t p(u)r(u)du\right)dt.$$

proof.

Observe that

$$\begin{aligned} A_0(t) &\equiv P\{X(t) = 1,\ t \le V_1\} \\ &= \sum_{r=1}^{\infty} P\{X(t) = 1,\ t \le V_1 | N_1 = r\} P\{N_1 = r\} \\ &= P\{T_{1,1} \ge t | N_1 = 1\} P\{N_1 = 1\} + \sum_{r=2}^{\infty} [P\{T_{1,1} \ge t | N_1 = r\} \\ &\quad + \sum_{j=1}^{r-1} P\{Z_{1,j} < t \le Z_{1,j} + T_{1,j+1} | N_1 = r\}] P\{N_1 = r\} \\ &= \overline{F}_{(1)1}(t) P\{N_1 = 1\} + \sum_{r=2}^{\infty} [\overline{F}_{(r)1}(t) \\ &\quad + \sum_{j=1}^{r-1} \int_0^t \overline{F}_{(r)j+1|Z_{1,j}=s}(t-s) dF_{(r)Z_{1,j}}(s)] P\{N_1 = r\}, \end{aligned} \tag{7}$$

where $\overline{F}_{(r)1}(t) = 1 - F_{(r)1}(t)$ and $\overline{F}_{(r)j+1|Z_{i,j}=s}(t)$ is the conditional survivor function of $T_{i,j+1}$ given $Z_{i,j} = s$ and $N_i = r$, that is, $P\{T_{i,j+1} \ge t | Z_{i,j} = s, N_i = r\}$, $i = 1, 2, \cdots$, $j = 1, 2, \cdots, r-1$, $r = 2, 3, \cdots$.

Note that, by the definition of $A(t)$, the following equation holds;

$$A(t) = A_{0(t)} + \int_0^t A(t-x) dH(x), \quad t \ge 0, \tag{8}$$

which is a renewal equation. Then the function $A(t)$ satisfies(see Theorem 4.1 of Karlin and Taylor(1975))

$$A(t) = A_0(t) + \int_0^t A_0(t-u) dM_H(u). \tag{9}$$

Note that, since $R_{i,j}$'s are finite with probability 1 and $\int_0^{\infty} p(u)r(u)du = \infty$(This means

$\sum_{j=1}^{N_i} T_{i,j}$ is finite with probability 1), V_1 is finite with probability 1. Then

$$\lim_{t\to\infty} A_0(t) \le \lim_{t\to\infty} P\{V_1 \ge t\} = 0,$$

and by the Key Renewal Theorem,

$$\begin{aligned}\lim_{t\to\infty} A(t) &= \lim_{t\to\infty} \int_0^t A_0(t-u)dM_H(u) \\ &= \frac{1}{E(Z_i)} \int_0^\infty A_0(t)dt.\end{aligned}$$

By noting that $\mu_{N-1} \equiv E[N_i - 1] = \int_0^\infty q(t)r(t)\exp\left(-\int_0^t p(u)r(u)du\right)dt$ (see Iyer(1992)), $E(Z_i)$ is obviously given by

$$E(Z_i) = \mu(p) + \mu_{N-1} \cdot \nu_1 + \nu_2,$$

where $\mu(p) = \int_0^\infty \exp\left(-\int_0^t p(u)r(u)du\right)dt$. On the other hand,

$$\begin{aligned}&\int_0^\infty A_0(t)dt \\ &= \int_0^\infty \overline{F}_{(1)1}(t)dt \cdot P\{N_1 = 1\} + \sum_{r=2}^{\infty}\Big[\int_0^\infty \overline{F}_{(r)1}(t)dt \\ &\qquad + \sum_{j=1}^{r-1}\int_0^\infty \int_0^t \overline{F}_{(r)j+1|Z_{1,j}=s}(t-s)dF_{(r)Z_{1,j}}(s)dt\Big] \cdot P\{N_1 = r\} \\ &= \mu_{(1)1} \cdot P\{N_1 = 1\} + \sum_{r=2}^{\infty}\{\mu_{(r)1} + \sum_{j=2}^{r}\mu_{(r)j}\} \cdot P\{N_1 = r\} \\ &= \sum_{r=1}^{\infty} E\Big(\sum_{j=1}^{r} T_{1,j} \mid N_1 = r\Big) \cdot P\{N_1 = r\} \\ &= \mu(p),\end{aligned}$$

where

$$\int_0^\infty \int_0^t \overline{F}_{(r)j+1|Z_{1,j}=s}(t-s)dF_{(r)Z_{1,j}}(s)dt = \mu_{(r)j+1}$$

by exchanging the order of integrals. This completes the proof.

Remark. Consider a coherent system consisting of n components. Assume that the n components operate independently of one another. Specifically, while a repair or a replacement of a failed component is occurring in one position, the other components continue to operate. Let h be the reliability function of the system. Then the steady state availability of the coherent system is given by

$$A = h(A_1, A_2, \cdots, A_n), \tag{10}$$

where A_i is the steady state availability of the component i (See Barlow and Proschan, 1975, pp. 192-193). In this situation, the condition of the existence of the steady state availability of each component is essential.

References

[1]. Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing, Probability Models*, Holt, Rinehart and Winston, New York.

[2]. Birolini, A. (1985). *On the Use of Stochastic Processes in Modeling Reliability Problems*, Springer-Verlag, New York.

[3]. Birolini, A. (1994). *Quality and Reliability of Technical Systems*, Springer-Verlag, New York.

[4]. Block, H. W., Borges, W. S. and Savits, T. H. (1985). Age-dependent Minimal Repair, *Journal of Applied Probability*, 22, 370-385.

[5]. Brown, M. and Proschan, F. (1983). Imperfect Repair, *Journal of Applied Probability,* 20, 851-859.

[6]. Hoyland, A. and Rausand, M. (1994). *System Reliability Theory : Models and Statistical Methods*, John Wiley & Sons, New York.

[7]. Iyer, S. (1992). Availability Results for Imperfect Repair, *Sankhya*, Series B, 54, 249-256.

[8]. Karlin, S. and Taylor, H. M. (1975). *A First Course in Stochastic Processes*, 2nd Ed., Academic Press, New York.

[9]. Mi, J. (1998). Some Comparison Results of System Availability, *Operations Research*, 45, 205-218.