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Porous Medium Theory in Consolidation

YOUNG-KYO SEO*
*Division of Ocean Development Engineering, Korea Maritime University, Busan, Korea

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요 약: 다공체 이론은 간극수압 및 토질입자 및 간극수의 상호 작용을 포함하는 여러 가지 지반관련 문제의 이해에 있어 매우 중요하다. 이러한 상호작용은 토질강도 및 변형에 중요한 영향을 미친다. 본 논문은 다공체 이론(porous medium theory)의 일반식 및 구성모델을 제시하고 그에 따른 유한요소 공식을 유도하였다. 압밀 예제로서 이러한 모델의 정확도를 검증하였다.

1. Introduction

Porous materials such as soils consist of a solid skeleton and voids or porosity which can contains various fluid and air. When loads are applied to the porous medium, there is an interaction between the deformation of the soil skeleton and fluid flow. Although the special cases of no-flow (undrained) and free-flow (drained) responses of porous media can be analyzed by single phase continuum formulations, it is generally essential to use two phase formulations to describe the effective stresses and pore-fluid pressures for a saturated porous material. Such theories were first developed by Terzaghi(1943) and Biot(1955) for linear elastic and linear visco-elastic porous materials. Some applications of the finite element method to the theory of elasto-plastic mixtures have been reported. Among them, Prevost (1980, 1982) and Borja (1986) developed the velocity(pressure)-displacement formulation of fluid saturated soil mixtures. While both formulations have their own advantages, a porous medium treatment with a velocity formulation is utilized in this work since it leads to homogeneous systems of finite element equations. The intent of this work is that introduction of such porous theory and the application into consolidation to account for a coupled pore-pressure and effective stress. In the following, the basic theory, as well as a numerical implementation and its consolidation examples are presented.

서영교 연락처: 부산광역시 영도구 동삼2동
051-410-4683 yseo@hhu.ac.kr

2. Theory of Porous Medium

2.1 Field equations

The treatment of soils to be employed here is that of a porous, granular solid skeletal continuum interacting with a continuous pore fluid. For clarity and completeness of the consolidation analysis framework, the basic mass and momentum balance equations for both fluid and solid phases of the soil are briefly developed below. A more extensive development of these equations can be found in Provest(1980). In the following, the average intrinsic micro densities of both the fluid and solid phases are denoted by ρ_α , where $\alpha = w$ denotes the fluid phase and $\alpha = s$ are the solid phase. In a representative volume element of soil, the respective volume fractions of the fluid and solid grain phases are denoted by n^w and n^s. Accordingly, macroscopic mass densities of the fluid and solid phases are denoted by ρ^w and ρ^s, and are related to the intrinsic average micro densities as follows

$$\rho^w = n^w \rho_w \quad \rho^s = n^s \rho_s \tag{1}$$

For the case of fully saturated soils, the macroscopic or bulk mass density ρ of the medium thus can be expressed as

$$\rho = \rho^s + \rho^w = n^s \rho_s + n^w \rho^w = 1 \tag{2}$$

The continuum mechanics sign convention is used in this work and so stresses and strains are positive in tension and negative in compression. Fluid pressures, however, are taken as positive in compression. With this sign convention, the balance of linear momentum equations for the fluid and solid phases can be expressed in general form as

$$\nabla \cdot \sigma^{\alpha} + \hat{p}^{\alpha} = \rho^{\alpha}(a^{\alpha} - b) \tag{3}$$

where $\hat{p}^{\alpha}$ is a momentum supply to the α^{th} constituent from the rest of the mixture due to interaction effects, and b is a body force per unit mass. Momentum transfer between the solid skeleton and the pore fluid is assumed to consist of diffusive and dilatational contributions as follows

$$\hat{p}^{s} = -\hat{p}^{w} = -\xi \cdot (v^{s} - v^{w}) - p_{w} \nabla n^{w} \tag{4}$$

where ξ is the soils' resistivity tensor which is merely the inverse of its symmetric, positive definite permeability tensor. In the general momentum balance equations (4), the partial stress tensor σ^{w} for the pore fluid is simply

$$\sigma^{w} = n^{w}\sigma_{w} = -n^{w} p_{w} \mathbf{1} \tag{5}$$

where p_{w} represents an average pore fluid pressure on the microscale, and $\mathbf{1}$ is the identity tensor. In a similar fashion, the mathematical expression for the partial stress tensor of the solid phase σ^{s} is

$$\sigma^{s} = n^{s}\sigma_{s} \tag{6}$$

where σ_{s} represents an average solid stress state in the soil on the microscale. The partial solid stress σ^{s} is not to be confused with Terzaghi's effective stress σ' (Terzaghi, 1943), although for most soil's the relationship between the two is straightforward. For example, the total average stress on a planar segment passing through a sequence of vanishingly small grain-to-grain contact areas can be written as

$$\sigma = \sigma^{s} + \sigma^{w} = \sigma' - p_{w}\mathbf{1} \tag{7}$$

Figure 1 shows the total and effective stresses in the porous media.

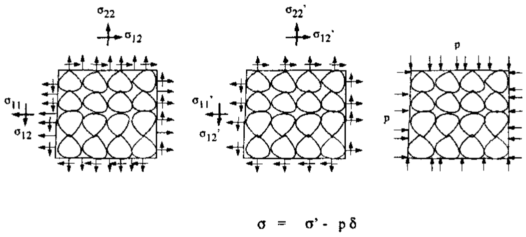


Fig. 1 *Total and effective stresses in porous medium*

When these notations mentioned above are employed in equation (3), and convective inertial term are neglected and under quasi-static conditions then specific linear momentum equations for both the skeleton and the pore fluid reduced to

$$\nabla \cdot (\sigma' - n^{s} p_{w}\mathbf{1}) - \xi \cdot (v^{s} - v^{w}) + \rho^{s} b = 0 \tag{8}$$

$$-\nabla(n^{w} p_{w}) + \xi \cdot (v^{s} - v^{w}) + \rho^{w} b = 0 \tag{9}$$

2.2 Continuum formulation

In the following, the response of soils subjected to the loading will be treated as general materially nonlinear parabolic initial boundary value problems in which the governing field equations are those provided in (8) and (9). With seepage and pore pressure effects included in these field equations, the analysis problems to be solved will feature physically based time dependence. Therefore, the strong form of the boundary value problems are as follows:

Find the soil displacement u^{s} and the pore fluid velocity v^{w} over the continuum soil domain Ω such that local equilibrium conditions (8) and (9) are satisfied throughout the soil mass subjected to the boundary conditions

$$v^{s} = \bar{v}^{s} \quad \text{on} \quad \Gamma_{g^{s}} \tag{10}$$

$$v^{w} = \bar{v}^{w} \quad \text{on} \quad \Gamma_{g^{w}} \tag{11}$$

$$n \cdot (\sigma' - n^{s} p_{w}\mathbf{1}) = \bar{h}^{s} \quad \text{on} \quad \Gamma_{h^{s}} \tag{12}$$

$$-n^{w} p_{w} n = \bar{h}^{w} \quad \text{on} \quad \Gamma_{h^{w}} \tag{13}$$

and the initial conditions are

$$u^{s}(x, 0) = u_{0}^{s}(x) \tag{14}$$

$$v^s(x,0)=v_0^s(x) \tag{15}$$

$$v^w(x,0)=v_0^w(x) \tag{16}$$

Part of the surface Γ, denoted by Γ_{g^s} and Γ_{g^w} are subjected to a prescribed displacement boundary conditions g, while the remainder of the surface Γ_{h^s} and Γ_{h^w} are subjected to tractions h. In equation (8), σ' represents the effective stress which is dependent upon the soil skeleton's constitutive properties and deformation history, succinctly expressed here as

$$\sigma=\sigma(\varepsilon, \dot{\varepsilon}, \zeta) \tag{17}$$

in which $\varepsilon=\nabla^s u$ denotes the skeletal small strain tensor and ζ represents a vector of internal state variables.

In soil mechanics, the pore fluid is often essentially incompressible since applied loads do not lead to appreciable changes in density and because the fluid is typically much less compressible than the soil skeleton. In such cases, the continuity equation imposes an excessive constraint, which causes mesh locking. In order to avoid such problems, special technique such as reduced and selective integration(Malkus and Hughes, 1978) and $\overline{B}$ method(Hughes, 1980) have been used incorporating the constraint into the constitutive equation. The rate type constitutive equation for the pressure is given by

$$\dot{p}=-\frac{\gamma^w}{n^w}\{n^s(\nabla \cdot v^s)+n^w(\nabla \cdot v^w)\} \tag{18}$$

2.3 Finite element formulation

Usage of a Galerkin weighted residual formulation in which the real and variational kinematic fields are expanded in terms of the same nodal basis functions, and discretization of the time domain into a finite number of discrete time points, leads to the following force balance equations at each unrestrained node A in the mesh of the soil domain as here at the $(n+1)^{th}$ time step:

$$(r_A)_{n+1}=(f_A^i)_{n+1}-(f_A^e)_{n+1}=0 \tag{19}$$

w h e r e

$$(f_A^i)_{n+1}=\begin{vmatrix} \int B_A^T(\sigma'-n^s p_w 1)_{n+1}\, d\Omega_s \\ -\int B_A^T(n^w p_w 1)_{n+1}\, d\Omega_s \end{vmatrix} \begin{vmatrix} -\int N_A\, \xi\, (v^s-v^w)_{n+1}\, d\Omega_s \\ \int N_A\, \xi\, (v^s-v^w)_{n+1}\, d\Omega_s \end{vmatrix} \tag{20}$$

$$(f_A^e)_{n+1}=\begin{bmatrix} \int N_A\, \rho^s\, b_{n+1}\, d\Omega_s & \int N_A \overline{h}^s_{n+1} d\Gamma_h \\ \int N_A\, \rho^w\, b_{n+1}\, d\Omega_s & \int N_A \overline{h}^w_{n+1} d\Gamma_h \end{bmatrix} \tag{21}$$

where B_A represents the nodal strain displacement matrix and N_A denotes the shape function for the A^{th} node. The quantity $(f_A^i)_{n+1}$ represents the internal forces (both solid and fluid) on node A at time t_{n+1} due to stresses in the soil mass, and $(f_A^e)_{n+1}$ represents the external forces applied to node A at time t_{n+1} due to body force and traction type loads. As long as balance can be achieved between the internal soil stresses and external forces, then the solutions to equation (19) will exist.

In general equation (11) represents a set of nonlinear algebraic equations which must be solved in an iterative fashion for the nodal velocities v_{n+1} at each time step. To obtain updated nodal displacements u_{n+1}, a generalized mid-point rule algorithm is used as

$$u_{n+1}=u_n+(1-\gamma)(\triangle t)_n\, v_n+\gamma(\triangle t)_{n+1}\, v_{n+1} \tag{22}$$

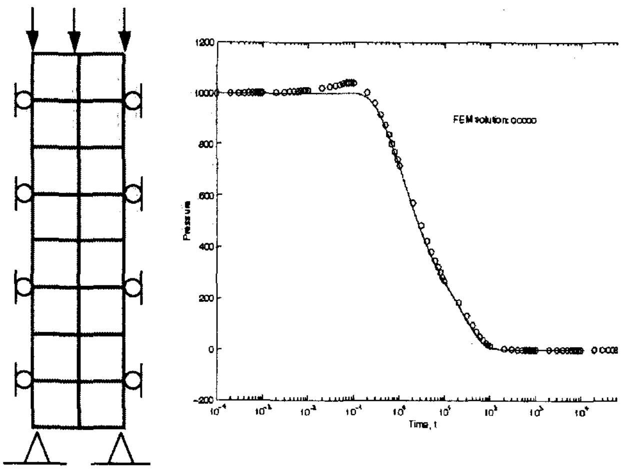
where $\gamma\in[0,1]$ is a constant integration parameter whose value is chosen as unity in the computations presented herein, while u_{n+1} and v_{n+1} denote, respectively, the displacement and velocity fields at time t_{n+1}

3. Consolidation Problems

The following examples demonstrated the accuracy of the foregoing numerical procedures which have been implemented in finite element code. The same quadrilateral elements were used with four nodes interpolating the displacements field for both solid and fluid. Standard Guassian quadrature rules were employed in the numerical integration, ie, 2×2 rule on the element stiffness. For the fluid phase reduced quadrature was used using a $\overline{B}$ procedure.

3.1 One-dimensional elastic consolidation

When the soil mass is subjected to a stress increase, the pore water pressure is suddenly increased. The excess pore water pressure generated due to loading gradually dissipates over a long period of time (that is, the consolidation). In order to simulate one-dimensional consolidation behavior, an uniform stripe load $p=1000N/m^2$ over the entire top surface was applied at time $t=0$, then held constant. Figure 2-a) shows the problem description for one-dimensional porous medium of elastic surface. The total initial height $H=8m$ and two columns of 16 elements are used. Each element has a side length of $1m$; Young's modulus $E=1.0\times10^7N/m^2$; Poisson's ratio $\nu=0.0$; coefficient of consolidation $c_v=1.0m^2/s$; the solid density $\rho^s=2.0\times10^3kg/m^3$; the fluid density $\rho^w=1.0\times10^3kg/m^3$; permeability $k=9.81\times10^{-4}m/s$; porosity $n^w=0.3$; bulk modulus of fluid $\lambda^w=2.0\times10^9N/m^2$. The same quadrilateral element were used with four nodes interpolating the displacements field for both solid and fluid. Standard Gaussian quadrature rules were employed in the numerical integration. For the fluid phase reduced quadrature was used using a $\overline{B}$ procedure.



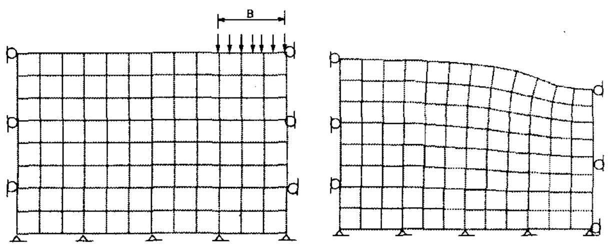
(a) Geometry (b) Pore water pressure

Fig. 2 One-dimensional consolidation geometry and pore pressure at depth 1.5m beneath the load

The analytical solution for pore water pressure is available for this problem (Das, 1994). A plot of u versus t is shown in Figure 2-b). Excellent agreement between the numerical and analytical results can be observed.

3.2 Two-dimensional elastic consolidation

Figure 3-a) shows the problem description for consolidation of an infinite elastic half surface. The initial height and width of the half surface are $8m$ and $12m$, respectively. The model was meshed with 96 elements. An uniform stripe load of with $2B$ with uniform intensity $p_w=1000N/m^2$ was applied on the top surface at time $t=0$ causing an initial fluid pressure $(i.e, p_w(0)=0.712P)$ (Chen, 1966) and thereafter held constant with drainage occurring at top surface. The same material propertied used in one dimensional problem are also employed in this problem. Figure 3-b) shows the deformed shapes of the soil mass. A plot of p_w versus t is shown in Figure 4.



(a) Undeformed (b) Deformed

Fig. 3 Two demensional consolidation problem

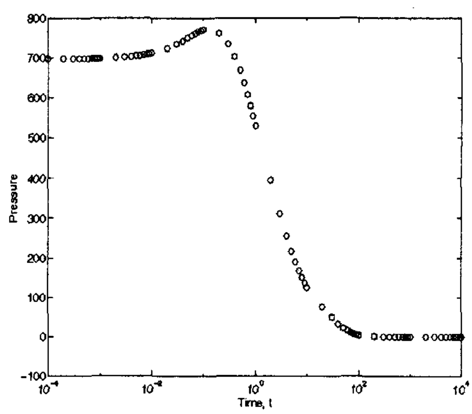


Fig. 4 Porewater pressure at depth z=1.5m beneath the applied strip load

5. Summary

In the consolidation analysis presented, the porous medium theory was introduced on a partially saturated soil deposit. It was found that the coupling between the soil skeletons' shear and compressibility behaviors are very important factors. On the demonstrated example, one and two dimensional consolidation example was demonstrated.

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