Hamiltonian Connectedness of Mesh Networks
with Two Wraparound Edges

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Abstract: An interconnection network is called *hamiltonian-connected* if there exists a hamiltonian path joining every pair of nodes. We consider the problem of adding edges to a mesh to make it hamiltonian-connected. We show that at least two edges are necessary for the problem. Also, we present the method to add two edges to a mesh so that the resulting network is hamiltonian-connected.

1. Introduction

In an interconnection network, one of the important feature is whether the network has a hamiltonian cycle or path. A *path(cycle)* in a graph G is called *hamiltonian* if it contains every vertex of G exactly once. G is called *hamiltonian-connected* if there exists a hamiltonian path joining every pair of vertices in G. If an interconnection network is hamiltonian-connected, the longest linear array from any node to any other node can be implemented in the network. There have been a lot of research results on whether an interconnection network has a hamiltonian path or cycle[2], [3], [6].

Meshes and tori represent the communication structures of many applications in scientific computations as well as the topologies of many large-scale interconnection networks. There is a polynomial-time algorithm for the hamiltonian path problem in a mesh, but mesh is not hamiltonian-connected because it is bipartite[8], [9]. Recently, Park and Kim[5] proved that $P_m \times C_n$ is hamiltonian-connected. $P_m \times C_n$ is a graph which has m wraparound edges in every row of an $m \times n$ mesh.

In this paper, we consider the minimum number of edges which should be added to a mesh to make it hamiltonian-connected. We describe a necessary condition that at least two edges should be added to a mesh to make it hamiltonian-connected. The condition is derived from the properties of mesh: it is bipartite and its minimum degree is two. Then, we propose two graphs, $G_1(m,n)$ and $G_2(m,n)$. They are obtained by adding one edge and two edges respectively between the same colored corner vertices in an $m \times n$ mesh. $G_2(m,n)$ is the only one which is satisfying the condition. If n is odd, then $G_2(m,n)$ has two wraparound edges in the first row

and the last row of an $m \times n$ mesh. It is a spanning subgraph of many interconnection networks such as tori, hypercubes, k-ary n-cubes and recursive circulants[6]. We show the hamiltonian properties of $G_1(m,n)$. And, we prove that $G_2(m,n)$ is hamiltonian-connected using the hamiltonian properties of $G_1(m,n)$. Thus, we show the minimum number of edges which should be added to a mesh to make it hamiltonian-connected is two.

The rest of the paper is organized as follows: In next section, some necessary definitions and notations are introduced. In Section 3, we show the hamiltonian properties of $G_1(m,n)$. We prove that $G_2(m,n)$ is hamiltonian-connected in Section 4. Finally, Section 5 concludes this paper.

2. Definitions and notation

Let $G = (V, E)$ be an $m \times n$ mesh, $V = \{v_j^i | 1 \le i \le m, 1 \le j \le n\}$ is the set of vertices(nodes), and $E = \{(v_j^i, v_{j+1}^i) | 1 \le i \le m, 1 \le j < n\} \cup \{(v_j^i, v_j^{i+1}) | 1 \le i < m, 1 \le j \le n\}$ is the set of edges(links).

A mesh G is bipartite, that is, the vertices can be colored with black and white in such a way that endvertices of every edge have different colors each other. v_j^i is called a *black* vertex if $i + j$ is even; otherwise, it is a *white* vertex. We let B and W are the sets of black and white vertices in G, respectively. We call a vertex in a mesh a *corner vertex* if it is of degree two.

We can consider that additional edges should be added to a mesh to make it hamiltonian-connected using the above property as follows:

- If mn is even, then $|B| = |W|$. A hamiltonian path joining every pair of white vertices has to contain an edge between a pair of black vertices. Similarly, a hamiltonian path joining every pair of black vertices has to contain an edge between a pair of white vertices. Thus, it is necessary to add at least two additional edges between two pair of vertices which have the same color each other.

- If mn is odd, then $|B| = |W| + 1$. A hamiltonian path joining every pair of black and white vertices has to contain an edge between a pair of black vertices. And, a hamiltonian path joining every pair of white

vertices has to contain two edges between two pair of black vertices. Thus, it is necessary to add at least two additional edges between two pair of black vertices.

Also, the minimum degree of a mesh G, $\delta(G)$, is two. In graph G, the vertices with degree 2 are four corner vertices. Thus, at least two edges are added between corner vertices so that $\delta(G) = 3$.

Now, we are going to propose a graph which is satisfying the above condition by adding two(minimum) edges to G. It is obtained by adding two edges between the same colored corner vertices in G.

Definition 1. Let $G = (V, E)$ is an $m \times n$ mesh.

(a) A graph $G_1(m, n)$ is defined as (V_{G_1}, E_{G_1}), where $V_{G_1} = V$, and $E_{G_1} = E \cup \{(v_1^1, v_n^1)\}$.

(b) A graph $G_2(m, n)$ is defined as (V_{G_2}, E_{G_2}), where $V_{G_2} = V$, and $E_{G_2} = E \cup \{(v_1^1, v_n^1), (v_1^m, v_n^m)\}$ if n is odd; otherwise, $E_{G_2} = E \cup \{(v_1^1, v_m^n), (v_m^1, v_1^m)\}$.

Without loss of generality, when either m or n is odd we assume n is odd. If n is even, then $G_2(m, n)$ has two additional edges between a pair of black corner vertices and a pair of white corner vertices(see Figure. 1(b)). If n is odd, then $G_2(m, n)$ has two additional edges in the first row and the last row of an $m \times n$ grid graph as follows: (a) When m is odd, $G_2(m, n)$ has two additional edges between two pair of black corner vertices(see Figure. 1(c)). (b) When m is even, $G_2(m, n)$ has two additional edges between a pair of black corner vertices and a pair of white corner vertices(see Figure. 1(d)). Thus, $G_2(m, n)$ is only one which is satisfying the above condition by adding two edges.

Also, it is a spanning subgraph of tori, hypercubes, k-ary n-cubes. $G_1(m, n)$ with odd n is a spanning subgraph of $G_2(m, n)$ (see Figure. 1(a)), and we will use its hamiltonian properties to prove hamiltonian-
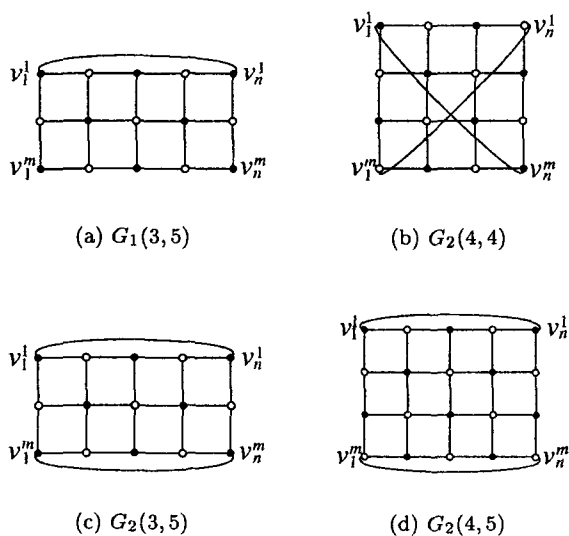
connectedness of $G_2(m, n)$.

We denote by $R(i)$ and $C(j)$ the vertices in row i and column j, respectively. That is, $R(i) = \{v_j^i | 1 \le j \le n\}$ and $C(j) = \{v_j^i | 1 \le i \le m\}$. We let $R(i : j) = \cup_{i \le k \le j} R(k)$ if $i \le j$; otherwise, $R(i : j) = \emptyset$. Similarly, $C(i : j) = \cup_{i \le k \le j} C(k)$ if $i \le j$; otherwise, $C(i : j) = \emptyset$.

We empoly two lemmas on hamiltonian properties of a mesh in [1], [4] to show the hamiltonian properties of $G_1(m, n)$ and $G_2(m, n)$.

Lemma 1. Let G be an $m \times n$ mesh, $m, n \ge 2$.
(a) If mn is even, then G has a hamiltonian path from any corner vertex v to any other vertex with color different from v.
(b) If mn is odd, then G has a hamiltonian path from any corner vertex v to any other vertex with the same color as v.

Lemma 2. Let G be an $m \times n$ mesh, and two vertices s, t have different color each other.
(a) For $m, n \ge 4$ even, G has a hamiltonian path from s to t.
(b) For $m = 2$, $n \ge 3$, and $s, t \notin C(k)(1 < k < n)$, G has a hamiltonian path from s to t.

We denote by $H[s, t|X]$ a hamiltonian path from s to t in the subgraph $G \langle X \rangle$ induced by X, if any. A path is represented as a sequence of vertices. If X is an empty set, $H[s, t|X]$ is an empty sequence.

3. Hamiltonian properties of $G_1(m, n)$

In this section, we will show the three hamiltonian properties of $G_1(m, n)$. These are employed in proving the hamiltonian connectedness of $G_2(m, n)$.

Lemma 3. For $m \ge 2$, $n \ge 3$ odd, $G_1(m, n)$ has a hamiltonian path between any corner vertex s and any other vertex t.

proof. We assume that $s = v_1^m$.

Case 1 m is even. By Lemma 1, there exists a hamiltonian path from s to any other vertex t which has a different color from s. If t has the same color as s, then we can construct a hamiltonian path P as follows: When $t \in C(2 : n)$, $P = (v_1^m, v_1^{m-1}, \cdots, v_1^2, v_1^1, H[v_n^1, t|C(2 : n)])$ by Lemma 1. When $t \in C(1)$, $P = (v_1^m, v_2^m, \cdots, v_{n-1}^m, v_n^m, v_n^{m-1}, v_n^{m-2}, \cdots, v_n^1, H[v_1^1, t|R(1 : m-1) \cap C(1 : n-1)])$. Note that v_n^1 and v_1^1 have a different color from t.

Case 2 m is odd. By Lemma 1, there exists a hamiltonian path from s to any other vertex t which has the same color as s. If t has a different color from s, then we can construct a hamiltonian path P as follows: When $t \in C(2 : n)$, $P = (v_1^m, v_1^{m-1}, \cdots, v_1^2, v_1^1, H[v_n^1, t|C(2 : n)])$ by Lemma 1. When $t \in C(1)$, $P = (v_1^m, v_2^m, \cdots, v_{n-1}^m, v_n^m, v_n^{m-1}, v_n^{m-2}, \cdots, v_n^1, H[v_1^1, t|R(1 : m-1) \cap C(1 : n-1)])$. $\square$

(a) $G_1(3, 5)$ (b) $G_2(4, 4)$

(c) $G_2(3, 5)$ (d) $G_2(4, 5)$

Figure 1. Examples of $G_1(m, n)$ and $G_2(m, n)$

Lemma 4. For $m \geq 1$, $n \geq 3$ odd, if vertices s,t are on the same row, and adjacent to each other, then $G_1(m,n)$ has a hamiltonian path from s to t.

proof. We assume that $s = v_i^m$ and $t = v_{i+1}^m$.

Case 1 $m = 1$. $G_1(m,n)$ is isomorphic to a ring. Thus, there exists a hamiltonian path from s to t.

Case 2 $m = 2$. By Lemma 2, there exists a hamiltonian path from s to t.

Case 3 $m \geq 3$. We can construct a hamiltonian path $P = (s, v_{i-1}^m, v_{i-2}^m, \cdots, v_1^m, H[v_1^{m-1}, v_n^{m-1}|R(1 : m-1)], v_n^m, v_{n-1}^m, \cdots, v_{i+2}^m, t)$. The existence of $H[v_1^{m-1}, v_n^{m-1}|R(1:m-1)]$ is due to Lemma 3. $\square$

Lemma 5. For $m \geq 2$, $n \geq 3$ odd, if s is on the last row and has the same color as corner vertex, then $G_1(m,n)$ has a hamiltonian path from s to any other vertex t which is in the different column from s.

proof. We assume that $s = v_i^m$ for $1 < i < m$ and i is odd. We can construct a hamiltonian path P from s to t as follows:

Case 1 $t \in W$. $P = (H[s, v_1^1|C(1 : i)], H[v_n^1, t|C(i + 1 : n)])$. The existence of $H[s, v_1^1 | C(1 : i)]$ and $H[v_n^1, t | C(i+1 : n)]$ are due to Lemma 2.

Case 2 $t \in B$. $P = (H[s, v_i^1|C(1 : i)], H[v_{i+1}^1, t|C(i + 1 : n)])$. The existence of $H[s, v_i^1 | C(1 : i)]$ and $H[v_{i+1}^1, t | C(i+1 : n)]$ are due to Lemma 2. $\square$

4. Hamiltonian connectedness of $G_2(m,n)$

In this section, we will prove that $G_2(m,n)$ is hamiltonian-connected. First, we prove the case that n is odd using the hamiltonian properties of $G_1(m,n)$. Then, we will prove the case that n is even.

Let P and Q be two vertex-disjoint path $(a_1, a_2, \cdots, a_k)$ and $(b_1, b_2, \cdots, b_l)$ on a graph G, respectively, such that (a_i, b_1) and (a_{i+1}, b_l) are edges in G. If we replace (a_i, a_{i+1}) with (a_i, b_1) and (a_{i+1}, b_l), then P and Q are merged into a single path $(a_1, a_2, \cdots, a_i, b_1, b_2, \cdots, b_l, a_{i+1}, \cdots, a_k)$. We call such a replacement a *merge* of P and Q w.r.t. (a_i, b_1) and (a_{i+1}, b_l).

Theorem 1. For $m \geq 2$, $n \geq 3$ odd, $G_2(m,n)$ is hamiltonian-connected.

proof. Let $s = v_i^x$ and $t = v_j^y$ for $i \leq j$, $x \leq y$. When $m = 2$, $G_2(2,n)$ is isomorphic to $P_2 \times C_n$. Thus, we will consider the case that $m \geq 3$.

Case 1 s and t are on the same row.

Case 1.1 Both s and t have a different color from v_1^k.

Case 1.1.1 When $s, t \in R(1)$, we can construct a hamiltonian path $P' = (H[s, v_1^2|R(1 : 2) \cap C(1 : i)], H[v_n^1, t|R(1 : 2) \cap C(i + 1 : n)])$ in $G\langle R(1:2)\rangle$. P' passes through an edge (x,y) in $R(2)$. Let x' and y' be the vertices in $R(3)$ adjacent to x and y, respectively. There exists a hamiltonian path P'' between x' and y' in $G\langle R(3:m)\rangle$ by Lemma 5. By a merge of P' and P'' w.r.t (x,x') and (y,y'), we have a hamiltonian path between s and t.

Case 1.1.2 When $s, t \in R(2)$, we can construct a hamiltonian path $P = (H[s, v_1^2|R(1 : 2) \cap C(1 : i)], P', H[v_n^2, t|R(1 : 2) \cap C(i + 1 : n)])$, where $P' = H[v_1^3, v_n^3|R(3 : m)]$. If $m = 3$, then $P'' = (v_1^3, v_2^3, \cdots, v_{n-1}^3, v_n^3)$; otherwise, P' exists by Lemma 3.

Case 1.1.3 When $s, t \in R(k)$, $2 < k < m - 2$, we can construct a hamiltonian path $P = (s, v_{i-1}^k, \cdots, v_1^k, H[v_1^{k-1}, v_{i+1}^{k-1}|R(1 : k - 1)], v_{i+1}^k, v_{i+2}^k, \cdots, v_{j-1}^k, H[(v_{j-1}^{k+1}, v_n^{k+1}|R(k + 1 : m)], v_n^k, v_{n-1}^k, \cdots, v_{j+1}^k, t)$. When $k = m$, P is same as Case 1.1.1. When $k = m - 1$, P is same as Case 1.1.2.

Case 1.2 Either s or t has the same color as v_1^k. There exists a hamiltonian path P' between s and t in $G\langle R(1:k)\rangle$ by Lemma 5. P' passes through an edge (x,y) joining a pair of vertices in $R(k)$ since at least one vertex in $R(k)$is contained in P' as an intermediate vertex. Let x' and y' be the vertices in $R(k+1)$ adjacent to x and y, respectively. By a merge of P' and P'' w.r.t (x, x') and (y, y'), we have a hamiltonian path between s and t, where P'' is a hamiltonian path between x' and y' in $G\langle R(k + 1 : m)\rangle$. The existence of P'' is due to Lemma 5.

Case 2 s and t are on the different rows.

Case 2.1 $m = 3$ and $s \in R(1)$, $t \in R(3)$.

Case 2.1.1 When both s and t are in W, we choose s' and t' such that s' is in $R(2)$, adjacent to t', and t' is in $R(3)$, adjacent to t. There exists a hamiltonian path $P = (H[s, s'|R(1, 2)], H[t', t|R(3)])$. Existence of $H[s, s'|R(1,2)]$ and $H[t', t|R(3)]$ due to Lemma 5 and Lemma 4 respectively.

Case 2.1.2 When $s \in B$ or $t \in B$ and s, t are on the same column, there exists a hamiltonian path $P = H[s, t|C(i : n)]$ by Lemma 1. $G\langle C(1 : i - 1)\rangle$ has a hamiltonian cycle C, since $i - 1$ is even. P passes through an edge (x,y) in $C(i)$. Let x' and y' be the vertices in $C(i - 1)$ adjacent to x and y, respectively. By a merge of P and $C - (x', y')$ w.r.t (x, x') and (y, y'), we have a hamiltonian path.

Case 2.1.3 Otherwise, there exists a hamiltonian path by Lemma 5.

Case 2.2 $m \geq 3$, $s \in R(1)$ and $t \in R(2)$.

Case 2.2.1 When $s, t \in W$, there exists a hamiltonian path P' between s and t in $G\langle R(1:2)\rangle$ by Lemma 5. P' passes through an edge (x,y) in $R(2)$. Let x' and y' be the vertices in $R(3)$ adjacent to x and y, respectively. By a merge of P' and P'' w.r.t (x, x') and (y, y'), we have a hamiltonian path, where P'' is hamiltonian path between x' and y' in $G\langle R(3 : m)\rangle$. The existence of P'' due to Lemma 4.

Case 2.2.2 When $s \in W$ and $t \in B$, there exists a hamiltonian path P as follows: (a) If s and t are on the same column, then $P = (H[s, s'|R(1)], H[t', t|R(2 : m)])$, where s' and t' are adjacent to s and t respectively, and s' is adjacent to t'. (b) Otherwise, if m is even, then $P = (H[s, v_i^m|C(1 : i)], H[v_{i+1}^m, t|C(i + 1 : n)])$. If m is odd, then $P = (H[s, v_1^1|C(1 : i), H[v_n^1, t|C(i + 1 : n)])$.

Case 2.2.3 $s \in B$. If s and t are on the same column, then a hamiltonian path is same as Case 2.2.1(a); Otherwise, there exists a hamiltonian path by Lemma 5.

Case 2.3 Otherwise

We can construct a hamiltonian path $P = (H[s, v_k^l | R(1 : l)], H[v_k^{l+1} | R(l+1 : m)])$ if $x = 1$, then $l = 2$; otherwise, $l = x$, and if $t = v_n^{x+1}$, then $k \neq i, n$; otherwise, $k = n$. □

Theorem 2. For $m \geq 2$, $n \geq 4$ even, $G_2(m, n)$ is hamiltonian-connected.

proof. Let $s = v_i^x$, $t = v_j^y$ for $i \leq j$, $x \leq y$.

Case 1 Either $s \in B$ and $t \in W$ or $s \in W$ and $t \in B$.

If $m = 2$ and $s, t \in C(k)$ for $1 < k < n$, then we can construct a hamiltonian path P joining s and t as follows: $P = (s, v_{i-1}^1, \cdots, v_1^1, H[v_n^2, v_n^1 | C(i+1 : n)], v_1^2, \cdots, v_{j-1}^2, t)$; otherwise, there exists a hamiltonian path by Lemma 2.

Case 2 $s \in B$ and $t \in B$.

When $s, t \in C(k)$ for $1 \leq k \leq n$, $H[s, v_n^1 | R(1 : l)]$ and $H[v_1^m, t | R(l+1 : m)]$ form a hamiltonian path, where $l = 2$ if $x = 1$; otherwise, $l = x$. Note that v_n^1 and v_1^m have different colors from s and t, respectively. Otherwise, we can construct a hamiltonian path P as follows: If $s \in C(1)$, then $P = (s, v_1^{i+1}, \cdots, v_1^m, H[v_n^1, v_n^{x-1} | R(1 : x-1)], H[v_n^x, t | C(2 : n) \cap R(x : m)])$. If $s \in C(2 : n-2)$, then $P = (H[s, v_1^m | C(1 : i)], H[v_n^1, t | C(i+1 : n)])$. If $s \in C(n-1)$, then $P = (H[s, v_n^1 | C(i : n) \cap R(1 : x)], H[v_1^m, v_{i-1}^m | C(1 : i-1)], H[v_i^m, t | C(i : n) \cap R(x+1 : m)])$.

Case 3 $s \in W$ and $t \in W$.

When $s, t \in C(k)(1 \leq k \leq n)$, we can construct a hamiltonian path $P' = (H[s, v_1^1 | R(1 : l)], H[v_n^m, t | R(l+1 : m)])$, where $l = 2$ if $x = 1$; otherwise, $l = x$. Similary, $s, t \in R(k)$ for $1 \leq k \leq m$, we can construct a hamiltonian path $P'' = (H[s, v_1^1 | C(1 : l)], H[v_n^m, t | C(l+1 : m)])$, where $l = 2$ if $i = 1$; otherwise, $l = i$. Otherwise, there exists a hamiltonian path as follows: If i is odd, a hamiltonian path is same as P'. If i is even, a hamiltonian path is same as P''. □

Corollary 1. The m-dimensional hypercube Q_m can be made to be hamiltonian-connected by adding two edges.

proof. Q_m has a $2^{\lfloor \frac{m}{2} \rfloor} \times 2^{\lceil \frac{m}{2} \rceil}$ mesh as a spanning subgraph[7]. By applying Theorem 2 to the mesh, the statement follows. □

5. Conclusion

In this paper, we showed the minimum number of edges which should be added to a mesh to make it hamiltonian-connected. We described a necessary condition that at least two edges should be added to a mesh to make it hamiltonian-connected, and proposed a graph which satisfys the condition. We proved that the proposed graph is hamiltonian-connected. Thus, we showed that a mesh can be made to be hamiltonian connected by adding two edges. Also, our result can be applied to other interconnection networks, such as tori, hypercubes, and k-ary n-cubes, which not are hamiltonian-connected but have a mesh as a spanning subgraph.

References

[1] C. C. Chen and N. F. Quimpo, "On strongly Hamiltonian abelian group graphs," *Combinatorial Mathematics VIII. Lecture Notes in Mathematics*, Vol. 884, pp. 23–34, 1981.

[2] S. J. Curran and J. A. Gallian, "Hamiltonian cycles and paths in Cayley graphs and digraphs - a survey," *Discrete Mathematics*, Vol. 156, pp. 1–18, 1996.

[3] S.-Y. Hsieh, G.-H. Chen and C.-W. Ho, "Hamiltonian-laceability of star graphs," *Networks*, Vol. 36, No. 4, pp. 225–232, 2000.

[4] A. Itai, C. H. Papadimitriou and J. L. Czwarcfiter, "Hamiltonian paths in grid graphs," *SIAM Journal of Computing*, Vol. 11, No. 4, pp. 676–686, 1982.

[5] J.-H. Park and H.-C. Kim, "Fault Hamiltonicity of product graph of path and cycle," submitted for publication.

[6] J.-H. Park and K.-Y. Chwa, "Recursive circulants and their embeddings among hypercubes," *Theoretical Computer Science*, Vol. 224, pp. 35–62, 2000.

[7] Y. Saad and M. H. Schultz, "Topological Properties of Hypercubes," *IEEE Transactions on Computers*, Vol. 37, No. 7, pp. 867–872, 1988.

[8] C. Umans and W. Lenhart, "Hamiltonian cycles in solid grid graphs," *In Proc. 38th Annual Symposium, FOCS '97*, pp. 496–507, 1997.

[9] C. Zamrescu and T. Zamrescu, "Hamiltonian properties of grid graphs," *SIAM Journal of Discrete Mathematics*, Vol. 5, pp. 564–570, 1992.

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