Necessary and Sufficient Conditions for Characteristic Transfer Function Matrices

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Abstract: There exist several forms of transfer function descriptions for multivariable LTI systems. We treat transfer function matrix with characteristic polynomial as its common denominator named Characteristic Transfer-function Matrices (CTM). First, we clarify necessary and sufficient conditions of CTM, then, we show some related lemmas. These interpretations not only offer deeper explanations but they also provide ways for calculations of all possible transfer matrices, system zeros, and inverse polynomial matrices.

1. Introduction

Several forms of transfer function descriptions have been developed for multivariable linear time invariant (LTI) systems [1-6]. Coprime factorization of a transfer function matrix as polynomial or rational matrix fractional descriptions is comprehensive one and well used to analyze and/or design control systems. On the other hand, matrix with coprime rational elements is a most primitive description and is given straightforward way from set of scalar systems. Polynomial numerator matrix over least common denominator is a relevant but compact description.

This paper deals with description of a transfer-function matrix with characteristic polynomial as its common denominator named Characteristic Transfer-function Matrices (CTM) [7], which have 'nice' properties. We clarify necessary and sufficient conditions of CTM, and also show some corresponding lemmas. These results explain novel interpretation of transfer matrices and also provide ways for calculation of system zeros, inverse polynomial matrices.

2. Definition and advantages of CTM

Definition 1

An $m \times p$ numerator polynomial matrix over common denominator polynomial of order n, $G(s) = \dfrac{N(s)}{d(s)}$ is said to be *Characteristic Transfer-function Matrices* (CTM) of a p-input m-output nth-order LTI system if and only if it is given by Equation (1).

$$G(s) = \frac{C adj(sI - A)B + D\det(sI - A)}{\det(sI - A)} \tag{1}$$

Here A, B, C and D are, respectively, $n \times n$, $p \times n$, $m \times n$, $m \times p$ coefficient matrices with real constant elements of state equation about the LTI system.

CTM contains a characteristic polynomial explicitly, thus it has the following nice properties.

(1) The degree of a system, system zeros, system poles are derived straightforward way.

(2) Denominator is scalar thus inverse matrix is not used and calculation for connection of systems is easily done.

(3) Non-minimal order, non-invertible and singular system can be described without cancel of hidden modes.

(4) The description is naturally expansion of scalar system.

Instead, not all elements in numerator polynomial matrix $N(s)$ are settled arbitrarily even if all constant elements of 4-tuple $\{A, B, C, D\}$ are free, and realizable CTM is constrained.

In the following chapters, a theorem on realizable class of CTM and corresponding lemma for interpretation of CTM are shown.

3. Necessary and sufficient conditions of CTM

Theorem 1[7]

Suppose $N(s)$ be $m \times p$ polynomial matrix and let $d(s)$ be a polynomial. Then, $G(s) = \dfrac{N(s)}{d(s)}$ is a Characteristic Transfer-function Matrix (CTM) if and only if $G(s)$ satisfies the following conditions.

Condition 1 Every element in the $G(s)$ is a proper real rational function.

Condition 2 Every $k \times k$ minor of the $N(s)$ contains $d(s)^{k-1}$ or zero. Where, $k = 1, 2, \cdots, \rho$. $\rho = rank\, N(s)$

Corollary 1

When all eigenvalues of A are distinct, the condition 2 in theorem 1 can be replaced by an alternative condition 2a.

Condition 2a $\quad rank N(\lambda_i) \le 1$ for all i (3)

Here, λ_i : distinct poles that satisfy $d(s) = 0$

<Proof of Theorem 1>
The proof of this theorem is shown based on Jordan canonical form in [7]. We will newly provide simpler proof using Rosenbrock's 'system zero' [1,4].

(Necessity)
Condition 1 is needed for meeting causality.
Necessity of condition 2 is shown by contradiction method.

Suppose a system described as $G(s) = \frac{N(s)}{d(s)}$ can be realized in the form of $\{A, B, C, D\}$ where the relation: $d(s) = \det(sI - A)$ holds. Zeros of the k-input k-output invertible subsystem can be obtained from the zero polynomials $z_k(s)$ that satisfy,

$$z_k(s) = \gcd_{i,j}[\det\{{}_kP_{ij}(s)\}] = 0 \quad (k=1,2,\cdots,\rho) \tag{4}$$

Where ${}_kP_{ij}(s) = \begin{bmatrix} A - sI & B^j \\ C^i & D^{ij} \end{bmatrix}$. The $\begin{bmatrix} C^i & D^{ij} \end{bmatrix}$ and $\begin{bmatrix} B^{jT} & D^{ijT} \end{bmatrix}^T$ are, respectively, appropriate k rows of $\begin{bmatrix} C & D \end{bmatrix}$ and columns of $\begin{bmatrix} B^T & D^T \end{bmatrix}^T$, which make $\det\{{}_kP_{ij}(s)\} \ne 0$ for every $k=1,2,\cdots,\rho$.

Rewriting equation (2), we get the zeros of the subsystem as the following,

$$\begin{aligned} z_k(s) &= \gcd_{i,j}[\det(A - sI)\det\{D^{ij} + C^i(sI - A)^{-1}B^j\}] \\ &= \gcd_{i,j}[\det(A - sI)\det(\frac{N^{ij}(s)}{d(s)})] \\ &= \gcd_{i,j}[\det(A - sI)\frac{\det N^{ij}(s)}{d(s)^k}] \\ &= \frac{\gcd_{i,j}[\det N^{ij}(s)]}{d(s)^{k-1}} \end{aligned} \tag{5}$$

where, $N^{ij}(s)$ is submatrix of $N(s)$ consists of corresponding rows and columns with i and j mentioned in D^{ij}.

If condition 2 does not hold, not all $z_k(s)$ become polynomials.

(Sufficiency)
Assume the condition 1, there exists realizations of the $G(s)$. Additionally, we shall show if condition 2 is satisfied, there exists a realization of which $d(s) = \det(sI - A)$ among them. To clarify this fact, it is enough to show that the McMillan degree of the $G(s)$ is not greater than the degree of $d(s)$.

Let $G_k(s)$ be $k \times k$ minor of the $G(s)$, then $G_k(s)$ can be derived as

$$G_k(s) = \frac{\det N^{ijk}(s)}{d(s)^k} \tag{6}$$

From assumption (2), every $\det N^{ijk}(s)$ is divisible by $d(s)^{k-1}$ for $k = 1,2,\cdots,\rho$, then $d(s)$ can be a common denominator of all $G_k(s)$. Similarly, McMillan degree of the $G(s)$ is given as order of a least common denominator of all $G_k(s)$. Thus we see that McMillan degree of the $G(s)$ is not greater than the degree of $d(s)$

We have thus completed the proof of Theorem 1. ■

Example 1

About the following transfer function matrices, which have the same input−output properties, (*i*) is not CTM since this does not satisfy condition 2, but (*ii*) and (*iii*) are CTM.

(*i*) $\dfrac{\begin{bmatrix} 1 & 0 \\ 0 & (s-2) \end{bmatrix}}{(s-1)(s-2)}$, (*ii*) $\dfrac{\begin{bmatrix} s-1 & 0 \\ 0 & (s-1)(s-2) \end{bmatrix}}{(s-1)^2(s-2)}$,

(*iii*) $\dfrac{\begin{bmatrix} (s-1)(s-2) & 0 \\ 0 & (s-1)(s-2)^2 \end{bmatrix}}{(s-1)^2(s-2)^2}$

Example 2
Consider the system described by matrix with coprime rational elements as,

$$G(s) = \begin{bmatrix} \dfrac{(s-b)}{(s-a)} & \dfrac{1}{(s-c)} \\ \dfrac{1}{(s-a)(s-b)(s-c)} & \dfrac{(s-a)}{(s-b)} \end{bmatrix}$$

its least-order CTM can be shown as

$$G(s) = \frac{\begin{bmatrix} (s-b)^2(s-c)^2 & (s-a)(s-b)(s-c) \\ (s-c) & (s-a)^2(s-c)^2 \end{bmatrix}}{(s-a)(s-b)(s-c)^2}$$

4. Related lemmas

Definition 2

Rosenbrock's system zeros of a realization $\{A, B, C, D\}$ for given CTM are called the system zeros of the CTM.

Lemma 1

System zeros of the $G(s)=\frac{N(s)}{d(s)}$ can be calculated as the solutions of the following equation.

$$\frac{\Delta_\rho(s)}{d(s)^{\rho-1}}=0 \tag{7}$$

Here, $\Delta_\rho(s)$ is ρ – th minor of $N(s)$ that is divisible by $d(s)^{\rho-1}$. $\rho = rank\ G(s)$.

<Proof> Obvious from the proof of Theorem 1.

Lemma 2

Inverse matrix of a regular polynomial matrix $L(s)$ satisfies condition 2 of Theorem 1 where $\det L(s)$ is $d(s)$. Furthermore, when $L(s)$ is row (column) proper, the $L(s)^{-1}$ also satisfies condition 1 of Theorem 1, then it becomes a CTM with $\det L(s)$ as its common denominator. Here, without loss of generality, let $\det L(s)$ be a monic polynomial.

<Proof>

An $L(s)$ can be transformed to Smith canonical form with adequate unimodular matrix $U_1(s), U_2(s)$ as

$$\begin{aligned} S(s) &= U_1(s)\cdot L(s)\cdot U_2(s) \\ &= diag(\mu_i(s)) \ ; i=1,2,\cdots,m \end{aligned} \tag{8}$$

where m is size of $L(s)$. Inverse matrix of $S(s)$ can be calculated as

$$S(s)^{-1} = \frac{diag(\frac{\det L(s)}{\mu_i(s)})}{\det L(s)} \tag{9}$$

Because of $\prod_{i=1}^{m}\mu_i(s)=\det L(s)$, it is obvious that $S(s)^{-1}$ becomes CTM with $\det L(s)$ as its common denominator. The $L(s)^{-1}$ also satisfy the condition 2 of Theorem 1, since $L(s)^{-1}=U_2(s)\cdot S(s)^{-1}\cdot U_1(s)$ and the condition 2 holds even though multiplied by unimodular matrices. Moreover, if $L(s)$ is row (column) proper, the $L(s)^{-1}$ becomes proper rational matrix then it also satisfies condition 1 of Theorem 1. ■

Lemma 3

Suppose a CTM: $G(s)=\frac{N(s)}{d(s)}$ satisfies $\det(\frac{N(\infty)}{d(\infty)})\neq 0$, then $G(s)^{-1}$ is also CTM and system zeros of $G(s)$ is system poles of $G(s)^{-1}$.

<Proof>

The proof is obvious in case m=1. When $m\geq 2$, $G(s)^{-1}$ can be derived as

$$\begin{aligned} G(s)^{-1} &= \frac{adjG(s)}{\det G(s)} \\ &= \frac{adjN(s)/d(s)^{m-2}}{\det N(s)/d(s)^{m-1}} \end{aligned} \tag{10}$$

Using the proofs of Theorem 1 and Lemma 1, we find that both $adjN(s)/d(s)^{m-2}$ and $\det N(s)/d(s)^{m-1}$ become polynomials after cancellations. By assumption of the theorem, we see that all elements in $G(s)^{-1}$ is proper. By Lemma 2, we recognize that $\frac{adjN(s)}{\det N(s)}$ is a CTM and then, $\frac{adjN(s)/d(s)^{m-2}}{\det N(s)/d(s)^{m-1}}$ satisfies the condition 2 of Theorem 1. All these things make it clear that $G(s)^{-1}$ can be expressed by CTM with system zeros of $G(s)$: $\det N(s)/d(s)^{m-1}$ as its common denominator. ■

5. Conclusion

Realizable class of characteristic transfer function matrices (CTM) has been discussed. Necessary and sufficient conditions, related corollary and lemmas were shown.

These results add new information to transfer function expression from novel point of view, and also provide ways for calculation of system zeros, inverse polynomial matrices. Furthermore, they will form the foundation for the control system design with fixed order controllers.

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