A SYMMETRIC-DEFINITE PENCIL APPROACH TO SOURCE SEPARATION

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ABSTRACT

A matrix pencil method for source separation [3] was shown to be an unbiased signal extractor in the presence of temporally white noise. Its efficiency and robustness lies in the fact that the method in [3] employs only time-delayed correlation matrices of the observation data. In this paper we stress out that the matrix pencil method might suffer from a numerical instability problem, because the symmetric-definite pencil was not exploited. Moreover we present a simple method of constructing a symmetric-definite pencil so that the matrix pencil method is numerically stable.

1. INTRODUCTION

Source separation is a fundamental problem that is encountered in many applications such as digital/wireless communications, signal/image processing, speech processing, and biomedical signal analysis where multiple sensors or multiple observation signals are involved. See [10] for more details on source separation. In the context of source separation, the m-dimensional observation vector $\boldsymbol{x}(t)$ is assumed to be generated by

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{v}(t), \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is the unknown mixing matrix, $\boldsymbol{s}(t)$ is the n-dimensional source vector and $\boldsymbol{v}(t)$ is additive white noise that is statistically independent of $\boldsymbol{s}(t)$. The task of source separation is to estimate the mixing matrix $\boldsymbol{A}$ (or its inverse that is referred to as a demixing matrix $\boldsymbol{W} = \boldsymbol{A}^{\#}$ where $\#$ is the pseudoinverse), given only a finite number of observation data $\{\boldsymbol{x}(t)\}$, $t = 1, \dots, N$. It is well known that there exist two indeterminacies such as ordering and scaling ambiguity. In other words, a feasible solution to source separation problem boils down to estimating a mixing matrix $\boldsymbol{A}$ such that a mapping from source vector to estimated source vector becomes transparent (i.e., $\boldsymbol{WA}$ becomes a generalized permutation matrix).

It is known that second-order statistics (SOS) is sufficient for successful source separation, provided sources have non-vanishing temporal correlations or the variances of sources are slowly time-varying (i.e., second-order nonstationary sources). Along this line, various SOS-based source separation methods have been developed. These include AMUSE [15], Molgedey-Schuster [13], SOBI [1], correlation matching [4, 7], matrix pencil method [3], and SEONS [8, 9].

Among these SOS-based source separation methods, a matrix pencil method [3] employs only two time-delayed correlation matrices to estimate the mixing matrix $\boldsymbol{A}$ in the presence of temporally white noise. In such a case, the matrix pencil method was shown to be an unbiased signal extractor. In fact, the matrix pencil method generalizes AMUSE where one equal-time correlation matrix and one time-delayed correlation matrix were used. Independently a similar idea was also proposed in [6, 5]. One critical problem that was not recognized in [3] is that the matrix pencil method might suffer from a numerical instability problem in the calculation of the generalized eigenvectors, because the pencil is not a symmetric-definite pencil. In general, a time-delayed correlation matrix is not positive definite, which might cause a serious problem in the task of source separation. In this paper, I stress out the importance of the symmetric-definite pencil and discuss a simple method of constructing a symmetric-definite pencil.

As in [3] we consider the data model (1) which satisfies the following assumptions:

- Source signals are assumed to be spatially uncorrelated but to have non-vanishing temporal correlations, i.e.,

$$E\{\boldsymbol{s}(t)\boldsymbol{s}^T(t-\tau)\} = \text{diag}\,\{\gamma_1(\tau), \dots, \gamma_n(\tau)\}, \tag{2}$$

  where E denotes the statistical expectation operator.

- Additive noises $\{v_i(t)\}$ are assumed to be spatially correlated but temporally white, i.e.,

$$E\{\boldsymbol{v}(t)\boldsymbol{v}(t-\tau)\} = \delta_\tau \boldsymbol{\Sigma}, \tag{3}$$

  where δ_τ is the Kronecker delta and $\boldsymbol{\Sigma}$ is an arbitrary $n \times n$ matrix.

2. SOS-BASED SOURCE SEPARATION

This section briefly reviews a main idea of SOS-based source separation methods. For the moment, the noise is assumed to be i.i.d. isotropic Gaussian process, i.e., the covariance matrix of noise vector $\boldsymbol{v}(t)$ has the form

$$\boldsymbol{R}_v(0) = E\{\boldsymbol{v}(t)\boldsymbol{v}^T(t)\} = \sigma_v^2 \boldsymbol{I}_m, \tag{4}$$

where E denotes the statistical expectation operator, $\boldsymbol{I}_m$ is the $m \times m$ identity matrix, and σ_v^2 is the noise variance.

The correlation matrices of the observation vector $\boldsymbol{x}(t)$ satisfy

$$\boldsymbol{R}_x(0) - \sigma_v^2 \boldsymbol{I}_m = \boldsymbol{A}\boldsymbol{R}_s(0)\boldsymbol{A}^T, \tag{5}$$

$$\boldsymbol{R}_x(\tau) = \boldsymbol{A}\boldsymbol{R}_s(\tau)\boldsymbol{A}^T, \tag{6}$$

for some non-zero time-lag τ and both $\boldsymbol{R}_s(0)$ and $\boldsymbol{R}_s(\tau)$ are diagonal matrices since sources are assumed to be spatially uncorrelated. In the case of overdetermined mixtures ($m > n$), the noise variance σ_v^2 can be estimated from the least singular value of $\boldsymbol{R}_x(0)$ (or the average of minor $m - n$ singular values of $\boldsymbol{R}_x(0)$). However such an estimate of noise variance is not reliable or is difficult to calculate when signal to noise ratio (SNR) is low or the variance of each sensor noise is different.

Denote $\widetilde{R}_x(0) = R_x(0) - \sigma_v^2 I_m$. Then the pseudo-inverse of the mixing matrix, $A^{\#}$ can be identified up to its re-scaled and permuted version by the simultaneous diagonalization of $\widetilde{R}_x(0)$ and $R_x(\tau)$, provided that $R_s^{-1}(0)R_s(\tau)$ has distinct diagonal elements. In fact, this is the main idea of AMUSE [15] that was motivated by FOBI [2]. This fundamental result is described in the following theorem

Theorem 1 *Let $\Lambda_1, D_1 \in \mathbb{R}^{n\times n}$ be diagonal matrices with positive diagonal entries and $\Lambda_2, D_2 \in \mathbb{R}^{n\times n}$ be diagonal matrices with non-zero diagonal entries. Suppose that $G \in \mathbb{R}^{n\times n}$ satisfies the following decompositions:*

$$D_1 = G\Lambda_1 G^T, \quad (7)$$
$$D_2 = G\Lambda_2 G^T. \quad (8)$$

Then the matrix G is the generalized permutation matrix, i.e., $G = P\Lambda$ if $D_1^{-1}D_2$ and $\Lambda_1^{-1}\Lambda_2$ have distinct diagonal entries.

Proof: See [9] for proof.

2.1. Simultaneous Diagonalization

In general we can find a linear transformation which simultaneously diagonalizes two symmetric matrices. For the sake of simplicity, the simultaneous diagonalization is explained in the case of $m = n$ and noise-free mixtures. Thus we deal with $R_x(0)$ and $R_x(\tau)$. The simultaneous diagonalization consists of two steps (whitening followed by an unitary transformation):

(1) First, the matrix $R_x(0)$ is whitened by

$$z(t) = D_1^{-\frac{1}{2}} U_1^T x(t), \quad (9)$$

where D_1 and U_1 are the eigenvalue and eigenvector matrices of $R_x(0)$ as

$$R_x(0) = U_1 D_1 U_1^T. \quad (10)$$

Then we have

$$R_z(0) = D_1^{-\frac{1}{2}} U_1^T R_x(0) U_1 D_1^{-\frac{1}{2}} = I_m,$$
$$R_z(\tau) = D_1^{-\frac{1}{2}} U_1^T R_x(\tau) U_1 D_1^{-\frac{1}{2}}.$$

(2) Second, a unitary transformation is applied to diagonalize the matrix $R_z(\tau)$. The eigen-decomposition of $R_z(\tau)$ has the form

$$R_z(\tau) = U_2 D_2 U_2^T. \quad (11)$$

Then $y(t) = U_2^T z(t)$ satisfies

$$R_y(0) = U_2^T R_z(0) U_2 = I_m,$$
$$R_y(\tau) = U_2^T R_z(\tau) U_2 = D_2.$$

Thus both matrices $R_x(0)$ and $R_x(\tau)$ are simultaneously diagonalized by a linear transform $W = U_2^T D_1^{-\frac{1}{2}} U_1^T$. It follows from Theorem 1 that $W = U_2^T D_1^{-\frac{1}{2}} U_1^T$ is a valid demixing matrix if all the diagonal elements of D_2 are distinct.

2.2. Generalized Eigenvalue Problem

The simultaneous diagonalization of two symmetric matrices can be carried out without going through two-step procedures. From the discussion in Section 2.1, we have

$$W R_x(0) W^T = I_m, \quad (12)$$
$$W R_x(\tau) W^T = D_2. \quad (13)$$

The linear transformation W which satisfies (12) and (13) is the eigenvector matrix of $R_x^{-1}(0)R_x(\tau)$ [11]. In other words, the matrix W is the generalized eigenvector matrix of the pencil $R_x(\tau) - \lambda R_x(0)$ [13].

3. SYMMETRIC-DEFINITE PENCILS

Recently Chang *et al.* proposed the matrix pencil method for BSS [3] where they exploited $R_x(\tau_1)$ and $R_x(\tau_2)$ for $\tau_1 \neq \tau_2 \neq 0$. Since the noise vector was assumed to be temporally white, two matrices $R_x(\tau_1)$ and $R_x(\tau_2)$ are not theoretically affected by the noise vector, i.e.,

$$R_x(\tau) = E\{x(t)x^T(t-\tau)\} = A R_s(\tau) A^T, \quad \text{for } \tau \neq 0. \quad (14)$$

Let us consider two different time-delayed correlation matrices (for $\tau_1 \neq \tau_2 \neq 0$)

$$R_1 = R_x(\tau_1) = A\Lambda_1 A^T, \quad (15)$$
$$R_2 = R_x(\tau_2) = A\Lambda_2 A^T, \quad (16)$$

where

$$\Lambda_1 = R_s(\tau_1) = \text{diag}\{\gamma_1(\tau_1), \ldots, \gamma_n(\tau_1)\}, \quad (17)$$
$$\Lambda_2 = R_s(\tau_2) = \text{diag}\{\gamma_1(\tau_2), \ldots, \gamma_n(\tau_2)\}. \quad (18)$$

It was shown in [3] that the demixing matrix $W = A^{-1}$ could be estimated by solving the generalized eigenvalue problem

$$R_2 U = R_1 U \text{diag}\{\lambda_1, \ldots, \lambda_n\}. \quad (19)$$

It leads to $W = U^T$, provided that $\left\{\lambda_i = \frac{\gamma_i(\tau_2)}{\gamma_i(\tau_1)}\right\}$ are distinct. Note that (19) is identical to the problem

$$R_1^{-1} R_2 U = U\Lambda, \quad (20)$$

which gives an LS-ESPRIT solution (see [3] for more details). This approach produces an estimate of the demixing matrix that is not sensitive to additive white noise and gives an closed-form solution. Unfortunately, however, this approach might have a numerical instability problem in the calculation of U since in general $R_2 - \lambda R_1$ is not a symmetric-definite pencil

The set of all matrices of the form $R_2 - \lambda R_1$ with $\lambda \in \mathbb{R}$ is said to be a *pencil*. Frequently we encounter into the case where R_2 is symmetric and R_1 is symmetric and positive definite. Pencils of this variety are referred to as *symmetric-definite pencils* [12].

Theorem 2 (pp. 468 in [12]) *If $R_2 - \lambda R_1$ is symmetric-definite, then there exists a nonsingular matrix $U = [u_1, \ldots, u_n]$ such that*

$$U^T R_1 U = \textit{diag}\{\gamma_1(\tau_1), \ldots, \gamma_n(\tau_1)\}, \quad (21)$$
$$U^T R_2 U = \textit{diag}\{\gamma_1(\tau_2), \ldots, \gamma_n(\tau_2)\}. \quad (22)$$

Moreover $R_2 u_i = \lambda_i R_1 u_i$ for $i = 1, \ldots, n$, and $\lambda_i = \frac{\gamma_i(\tau_2)}{\gamma_i(\tau_1)}$.

For the requirement of symmetry, we replace R_1 and R_2 by M_1 and M_2 that are defined by

$$M_1 = \frac{1}{2}\left\{R_1 + R_1^T\right\}, \tag{23}$$

$$M_2 = \frac{1}{2}\left\{R_2 + R_2^T\right\}. \tag{24}$$

Either M_1 or M_2 should be positive definite. Depending on the choice of the time-lag τ_1, the matrix M_1 can be negative definite. In order to avoid this, we consider a linear combination of time-delayed correlation matrices such that the combination is positive definite.

Denote

$$C_1 = \sum_{i=1}^{J} \alpha_i M_x(\tau_i), \tag{25}$$

for $J \geq 2$.

We find a set of coefficients, $\{\alpha_i\}$, such that the matrix C_1 is positive definite. Thus, the pencil $M_2 - \lambda C_1$ is symmetric-definite, so the generalized eigenvector matrix U that solves

$$M_2 U = C_1 U \text{diag}\{\lambda_1, \ldots, \lambda_n\}, \tag{26}$$

can be computed with a numerical problem. A simple modification of the matrix pencil method is done by replacing R_1 and R_2 by C_1 and M_2 for three methods (GED, LS-ESPRIT, TLS-ESPRIT) in [3]. A way of finding a set of coefficients, $\{\alpha_i\}$, such that the matrix C_1 is positive definite, is summarized below. In fact, this method was motivated by the finite step global convergence (FSGC) algorithm [14] that was originally developed for different purpose.

Algorithm Outline: Finding a positive definite C_1

1. Estimate time-delayed correlation matrices and construct an $m \times mJ$ matrix

$$\mathcal{M} = [M_x(\tau_1) \cdots M_x(\tau_J)]. \tag{27}$$

   Then compute the singular value decomposition (SVD) of $\mathcal{M}$, i.e.,

$$\mathcal{M} = U \Sigma V^T, \tag{28}$$

   where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{mJ \times mJ}$ are orthogonal matrices, and Σ has nonzero entries at (i,i) position ($i = 1, \ldots, n$) and zeros elsewhere. The number of sources, n can be detected by inspecting the singular values. Define U_s by

$$U_s = [u_1 \cdots u_n], \tag{29}$$

   where u_i is the ith column vector of the matrix U and $n \leq m$.

2. For $i = 1, \ldots, J$, compute

$$F_i = U_s^T M_x(\tau_i) U_s. \tag{30}$$

3. Choose any initial $\alpha = [\alpha_1 \cdots \alpha_J]^T$.

4. Compute

$$F = \sum_{i=1}^{J} \alpha_i F_i. \tag{31}$$

5. Compute a Schur decomposition of F and check if F is positive definite or not. If F is positive definite, the algorithm is terminated. Otherwise, go to Step 6.

6. Choose an eigenvector u corresponding to the smallest eigenvalue of F and update α via replacing α by $\alpha + \delta$ where

$$\delta = \frac{[u^T F_1 u \cdots u^T F_J u]^T}{\|[u^T F_1 u \cdots u^T F_J u]\|}. \tag{32}$$

   Go to step 4. This loop is terminated in a finite number of steps (see [14] for proof).

7. Compute

$$C = \sum_{i=1}^{J} \alpha_i M_x(\tau_i), \tag{33}$$

   and perform an eigenvalue-decomposition of C,

$$C = [U_{c1}, U_{c2}] \begin{bmatrix} D_1 & \\ & 0 \end{bmatrix} [U_{c1}, U_{c2}]^T \tag{34}$$

   where U_{c1} contains the eigenvectors associated with n principal singular values of D_1.

8. The robust whitening transformation is performed by

$$z(t) = Q x(t), \tag{35}$$

   where $Q = D_1^{-\frac{1}{2}} U_{c1}^T$.

Note: In the case of $m = n$ (equal number of sources and sensors), step 1 and 2 are not necessary. Simply we let $F_i = M_x(\tau_i)$.

Our algorithm, which is be called *"extended matrix pencil method"*, is summarized below.

Algorithm Outline: Extended Matrix Pencil Method

1. Compute $M_x(\tau_2)$ for some time-lag $\tau_2 \neq 0$ and calculate the matrix $C_1 = \sum_{i=1}^{J} \alpha_i M_x(\tau_i)$ by the FSGC method.

2. Find the generalized eigenvector matrix V of the pencil $M_x(\tau_2) - \lambda C_1$ which satisfies

$$M_x(\tau_2) V = C_1 V \Lambda. \tag{36}$$

3. The demixing matrix is given by $W = V^T$.

4. A NUMERICAL EXAMPLE

A numerical experiment was performed using MATLAB. We used three digitized voice signals and two music signals (all of which were sampled at 8 kHz), then generated 5-dimensional mixture vector by a randomly generated mixing matrix A. Spatially correlated but temporally white noise was added in the level of SNR 10 dB. In the matrix pencil method, we used $M_x(11)$ and $M_x(15)$ to estimate the mixing matrix. In such a case, the demixing matrix W using the matrix pencil method [3] came up with a complex-valued matrix due to a numerical instability problem. Two eigenvalues of $R_x^{-1}(11) R_x(15)$ were complex conjugate.

We applied the extended matrix pencil based on symmetric-definite pencil to the same data set. In order to construct a symmetric-definite pencil, we found a linear combination $C_1 = \sum_{i=1}^{11} \alpha_i M_x(i)$

such that C_1 is positive definite. We used C_1 and $M_x(15)$ to estimate the demixing matrix. Then the result was very successful, the global system matrix $G = WA$ was

$$G = \begin{bmatrix} -0.001 & 0.0052 & 0.009 & 0.018 & \mathbf{0.760} \\ -0.042 & 0.030 & -0.207 & \mathbf{1.113} & 0.011 \\ \mathbf{-0.406} & -0.011 & -0.010 & -0.001 & -0.003 \\ 0.021 & -0.106 & \mathbf{-1.317} & -0.178 & 0.003 \\ 0.010 & \mathbf{-0.301} & 0.029 & 0.018 & -0.001 \end{bmatrix}.$$

One can easily see that the matrix G has one dominant value in each row and column, which states that the separation is successful.

5. DISCUSSION

We have pointed out that the matrix pencil method [3] might suffer from a numerical instability problem because the generalized eigenvalue problem generally requires the symmetric-definite pencil. We have presented a simple method, *extended matrix pencil method*, where a symmetric-definite pencil was employed. Numerical experiments confirmed the high performance of the proposed method. The extended matrix pencil method can be also applied to blind separation of nonstationary sources.

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