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A note on measurable fuzzy mappings

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ABSTRACT

In this paper, we characterze the Borel σ-field generated by the Hausdorff-Skorokhod metric on the space of normal and upper-semicontinuous fuzzy sets with compact support in the Ecleadean space R^n. As a result, we give a characterization of measurable fuzzy mappings.

Keyword : The Hausdorff-Skorokhod metric, Measurable fuzzy mappings

1. Introduction

Let (Ω, Σ) be a measurable space. A multi-valued function $X : \Omega \hookrightarrow R^n$ is called measurable if for each closed subset C of R^n,

$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \cap C \neq \emptyset\} \in \Sigma$. It is known that if F is compact-valued, the measurability of a multi-valued function F is equivalent to the measurability of a function $X : \Omega \to (K(R^n), h)$, where $(K(R^n), h)$ is the metric space of all non-empty compact subsets of R^n endowed with the Hausdorff metric h.(See Castaing and Valadier [2], Klein and Thompson [9])

A fuzzy mapping $\tilde{X} : \Omega \hookrightarrow R^n$ is called measurable if for each closed subset C of R^n, the function $\tilde{X}^{-1}(B) : \Omega \to [0,1]$ defined by $\tilde{X}^{-1}(C)(\omega) = \sup_{x \in C} \tilde{X}(\omega)(x)$ is Σ-measurable. This definition of measurability for a fuzzy mapping was introduced by Butnariu [1]. Another concept of measurability for a fuzzy mapping can be introduced as suggested in Puri and Ralescu [10]. It turned out that alternative of measurability for a fuzzy mapping are equivalent. But it has been remained an open problem if, under what conditions, the measurability for a fuzzy mapping is equivalent to the measurability of a function $\tilde{X} : \Omega \to (F(R^n), d)$, where $(F(R^n), d)$ is an appropriate metric subspace of the space consisting of all normal fuzzy subsets of R^n.

In this paper, we show that if $F(R^n)$ is the space of all normal and upper-semicontinuous fuzzy sets in R^n with compact support, and $\mathfrak{X}$ is a $F(R^n)$-valued fuzzy mapping, then the measurability of $\mathfrak{X}$ is equivalent to the measurability of a function $\mathfrak{X}:\Omega \to (F(R^n), d_s)$, where d_s is the Hausdorff-Skorokhod metric on $F(R^n)$.

2. Preliminaries

Let $K(R^n)$ be the family of all non-empty compact subsets of R^n. Then $K(R^n)$ is metrizable by the Hausdorff metric h defined by

$$h(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\}$$

where $|\cdot|$ is the usual norm in R^n. It is well known that the metric space $(K(R^n), h)$ is complete and separable.(See Debreu [3])

Let $F(R^n)$ denote the space of all normal and upper-semicontinuous fuzzy sets u in R^n such that $supp\ u = cl\ \{x\in R^n : u(x)>0\}$ is compact. For a fuzzy set u in R^n, we define the α- level set of u by

$$[u]^{\alpha} = \begin{cases} \{x : u(x) \geq \alpha\}, & 0<\alpha\leq 1, \\ supp\ u, & \alpha = 0. \end{cases}$$

Then it follows that $u \in F(R^n)$ if and only if $[u]^{\alpha} \in K(R^p)$ for each $\alpha \in [0,1]$.

Lemma 2.1. For $u \in F(R^n)$, let us define $f_u:[0,1] \to ((K(R^n), h)$ by $f_u(\alpha) = [u]^{\alpha}$. Then (1) f_u is non-increasing; i.e., $\alpha \leq \beta$ implies $f_u(\alpha) \supset f_u(\beta)$.

(2) f_u is left-continuous on $(0,1]$.

(3) f_u has right-limits on $[0,1)$ and is right-continuous at 0.

Conversely, if $g:[0,1] \to ((K(R^n), h)$ is a function satisfying the above conditions (1)-(3), then there exists a unique $v \in F(R^n)$ such that $g(\alpha) = [v]^{\alpha}$ for all $\alpha \in [0,1]$.

Proof: See Kim [7].

If we denote the right-limit of f_u at $\alpha \in [0,1)$ by $f_u(\alpha^+)$, then

$$f_u(\alpha^+) = cl\{x\in R^n : u(x)>\alpha\}.$$

Now for $I \subset [0,1]$, if we define

$$w_u(I) = \sup_{\alpha_1,\alpha_2\in I} h(f_u(\alpha_1), f_u(\alpha_2)),$$

then it follows that for $0\leq\alpha<\beta\leq 1$,

$$w_u(\alpha,\beta) = w_u(\alpha,\beta] = h(f_u(\alpha^+), f_u(\beta)),$$

$$w_u[\alpha,\beta) = w_u[\alpha,\beta] = h(f_u(\alpha), f_u(\beta)).$$

Also, if we define $j_u(\alpha) = h(f_u(\alpha), f_u(\alpha^+))$, then the function f_u is continuous at α if and only if $j_u(\alpha) = 0$.

Lemma 2.2. For each $u \in F(R^n)$ and $\varepsilon > 0$, there exists a partition $0=\alpha_1<\alpha_2<\ldots<\alpha_r=1$ of $[0,1]$ such that $w_u(\alpha_{i-1}, \alpha_i] < \varepsilon$ for all $i=1,2,\ldots,r$.

Proof. See Joo and Kim [5].

The above lemma implies that $J(u) = \{\alpha : j_u(\alpha)>0\}$ is denumerable for each $u \in F(R^n)$.

3. Main Results

In this section, we show that if $\mathfrak{X}$ is a $F(R^n)$-valued fuzzy mapping, then the measurability of $\mathfrak{X}$ is equivalent to the measurability of a function $\mathfrak{X}:\Omega \rightarrow (F(R^n), d_s)$, where d_s is the Hausdorff-Skorokhod metric on $F(R^n)$.
First, in order to generalize the Hausdorff metric h on $K(R^n)$ to $F(R^n)$, we define the metric d_∞ on $F(R^n)$ by

$$d_\infty(u,v) = \sup_{0\le\alpha\le1} h([u]^\alpha,[v]^\alpha).$$

Then it is well-known that $(F(R^n),d_\infty)$ is complete, but is not separable. (See Klement et al. [8]) Recently, Joo and Kim [4, 5] introduced a new metric on $F(R^n)$ as follows:

Definition 3.1. Let T be the class of strictly increasing continuous mappings of $[0,1]$ onto itself. For $u,v \in F(R^n)$, we define

$$d_s(u,v) = \inf\{\varepsilon>0: \text{there exists a } t\in T \text{ s.t. } \sup_{0\le\alpha\le1}|t(\alpha)-\alpha| \le \varepsilon \text{ and } d_\infty(u,t(v)) \le \varepsilon\},$$

where $t(v)$ denotes the composition of v and t.

It follows immediately that d_s is a metric on $F(R^n)$ and $d_s(u,v) \le d_\infty(u,v)$. The metric d_s will be called the Hausdorff-Skorokhod metric. It is known that the metric space $(F(R^n),d_s)$ is separable and topological complete.

Lemma 3.2. Let us define

$L_\alpha : F(R^n) \rightarrow K(R^n)$ by $L_\alpha(u) = [u]^\alpha$.

Then (1) L_0 and L_1 are continuous.

(2) If $0<\alpha<1$, then L_α is continuous at u if and only if $\alpha \notin J(u)$, i.e., the function f_u defined in Lemma 2.1 is continuous at α.

Theorem 3.3. Let B_s be the Borel σ-field of $F(R^n)$ w.r.t. the Hausdorff-Skorokhod metric d_s. Then B_s coincides with the smallest σ-field of subsets of $F(R^n)$ for which the maps L_α are measurable for all $\alpha \in [0,1]$.

Corollary 3. 4. Let $\mathfrak{X}:\Omega \hookrightarrow R^n$ be a $F(R^n)$-valued fuzzy mapping. Then $\mathfrak{X}$ is measurable if and only if it is measurable when considered a function from Ω to the metric space $(F(R^n),d_s)$.

Corollary 3. 5. Let $F_C(R^n) = \{u\in F(R^n) : f_u \text{ is continuous on } [0,1]\}$. If $\mathfrak{X}:\Omega \hookrightarrow R^n$ is a $F_C(R^n)$-valued fuzzy mapping, then $\mathfrak{X}$ is measurable if and only if it is measurable when considered a function from Ω to the metric space $(F_C(R^n),d_\infty)$.

Remark. Kaleva [6] suggested that if $\mathfrak{X}$ is a measurable fuzzy mapping, then it is measurable when considered a function from Ω to the metric space $(F(R^n),d_\infty)$. But he proved only that the inverse image of each d_∞-open ball is measurable. Thus, his proof is incomplete, since the metric space $(F(R^n),d_\infty)$ is not separable. In fact, every d_∞-open ball is B_s-measurable because

$$\{u : d_\infty(u,v) < \delta\} = \bigcup_{0<\varepsilon<\delta}\bigcap_n\{u : h([u]^{\alpha_n},[v]^{\alpha_n})\},$$

where ε denotes rational numbers and $\{\alpha_n\}$ is an enumeration of all rational points in

$[0,1]$.

Let B_∞ be the Borel σ-field of $F(R^n)$ w.r.t. the metric d_∞. Since $d_s(u,v) \le d_\infty(u,v)$, it is clear that $B_s \subset B_\infty$. Now we show that $B_s \ne B_\infty$ and that there exists a $F(R^n)$-valued fuzzy mapping $\tilde{X}$ such that $\tilde{X}$ is B_s-measurable but is not B_∞-measurable. For $0<\lambda<1$, we let

$$u_\lambda(x) = \begin{cases} 1 & \text{if } x = 0, \\ \lambda & \text{if } 0<|x|\le 1. \\ 0 & elsewhere. \end{cases}$$

Then

$$[u_\lambda]^\alpha = \begin{cases} \{0\} & \text{if } \lambda<\alpha\le 1, \\ \{x : |x|\le 1\} & \text{if } 0\le\alpha\le\lambda. \end{cases}$$

Thus, $d_\infty(u_\lambda, u_\delta) = 1$ for $\lambda \ne \delta$. On the other hand, by rigorous process, we can obtain $d_s(u_\lambda, u_\delta) = |\lambda-\delta|$.

Let $F_0(R^n) = \{u_\lambda : 0<\lambda<1\}$. Then $(F_0(R^n), d_\infty)$ is a discrete space and so $B_\infty^0 = \{A\cap F_0(R^n): A\in B_\infty\}$ consists of all subsets of $F_0(R^n)$. Since $(F_0(R^n), d_s)$ can be identified with $(0,1)$ endowed with the usual metric, there exists a subset D of $F_0(R^n)$ such that

$$D \notin B_s^0 = \{A\cap F_0(R^n): A\in B_s\},$$

which implies $B_s \ne B_\infty$. Now if we let $\Omega = (0,1)$ and Σ be the the usual Borel σ-field of $(0,1)$, then the function $\tilde{X} : \Omega \to F_0(R^n)$ defined by $\tilde{X}(\omega) = u_\omega$ is B_s-measurable but is not B_∞-measurable because $\tilde{X}^{-1}(D) = \{\lambda : u_\lambda \in D\}$ is not Borel set in $(0,1)$.

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