퍼지 계수를 갖는 미분 시스템에 대한 퍼지 해의 존재성

The existence of the fuzzy solutions for the differential system with fuzzy coefficient

손기도*, 강점란**, 이부영***, 박용범****
부산시 사하구 하단동 동아대학교 수학과,
K. D. Son*, J. R. Kang**, B. Y. Lee***and Y. B. Park****
*Dept. of Mathematics, Dong-a University, Busan 604-714, Korea,
**Dept. of Mathematics, Dong-a University, Busan 604-714, Korea,
pointegg@hanmail.net,
***Dept. of Mathematics, Dong-a University, Busan 604-714, Korea,
bylee@daunet.donga.ac.kr.
****Division of Mathematical Sciences Pukyong National University,
Pusan 608-737, Korea
parkyb@pknu.ac.kr

ABSTRACT

In this paper, we study the existence of fuzzy solution for the following differential system with fuzzy coefficient using the different two methods:

$$\begin{cases} \dot{x}_1 = \tilde{a}\, x_2, \\ \dot{x}_2 = \tilde{b}\, x_1, \end{cases}$$

where $\tilde{a}$, $\tilde{b}$ is the fuzzy natural number generated by fuzzy number $\tilde{1}$. The α-level set of the fuzzy number $\tilde{1}$ is $[\tilde{1}]^\alpha = [\tilde{1}_l^\alpha, \tilde{1}_2^\alpha]$. The α-level set of $\tilde{a}$ is $[\tilde{a}]^\alpha = [a \cdot 1_l^\alpha, a \cdot 1_r^\alpha]$ and α-level set of $\tilde{b}$ is $[\tilde{b}]^\alpha = [b \cdot 1_l^\alpha, b \cdot 1_r^\alpha]$.

Keywords and Phrases : fuzzy number, fuzzy solution, α-level set, fuzzy process

1. Introduction

The concept of the natural number is easily extended to fuzzy case by max-min convolution. Let $\tilde{1}$ be a fuzzy number in R^+ with the following membership function.

$\forall x \in R^+$, $\mu_{\tilde{1}}(x) \in [0,1]$ and $\mu_{\tilde{1}}(1) = 1$.

We now proceed with the successive construction of fuzzy numbers as follows;

$$\tilde{2} = \tilde{1}(+)\tilde{1},\quad \tilde{3} = \tilde{2}(+)\tilde{1},\quad \cdots,$$
$$\tilde{n} = \widetilde{(n-1)}(+)\tilde{1}, \cdots.$$

Let us define the α-cut for the fuzzy number

$\tilde{1}$ as follows. $[\tilde{1}]^{\alpha}=[\tilde{1}_l^{\alpha}, \tilde{1}_r^{\alpha}]$, $\alpha\in[0,1]$.

It is constructed from 1 by the use of the intervals of confidence of level α.

$$[\tilde{2}]^{\alpha}=[\tilde{1}]^{\alpha}+[\tilde{1}]^{\alpha}=[2\cdot\tilde{1}_l^{\alpha}, 2\cdot\tilde{1}_r^{\alpha}] =2[\tilde{1}_l^{\alpha}, \tilde{1}_r^{\alpha}]$$

$$[\tilde{3}]^{\alpha}=[\tilde{2}]^{\alpha}+[\tilde{1}]^{\alpha}=[3\cdot\tilde{1}_l^{\alpha}, 3\cdot\tilde{1}_r^{\alpha}] =3[\tilde{1}_l^{\alpha}, \tilde{1}_r^{\alpha}],$$

$\cdots$

$$[\tilde{n}]^{\alpha}=[\widetilde{n-1}]^{\alpha}+[\tilde{1}]^{\alpha}=[n\cdot\tilde{1}_l^{\alpha}, n\cdot\tilde{1}_r^{\alpha}] =n[\tilde{1}_l^{\alpha}, \tilde{1}_r^{\alpha}].$$

We now call the fuzzy natural number generated by fuzzy number $\tilde{1}$. We define the multiplication by the fuzzy number for the real matrix. If $\tilde{a}$ is fuzzy number, $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$ where $a_{ij}\in R$ then $\tilde{a}\cdot A=\begin{pmatrix} \tilde{a}\cdot a_{11} & \tilde{a}\cdot a_{12} \\ \tilde{a}\cdot a_{21} & \tilde{a}\cdot a_{22}\end{pmatrix}$

In this paper we consider the solution of following differential equations with fuzzy coefficients using two different methods

$$\text{(F.D.E.)}\qquad \begin{cases} \dot{x}_1=\tilde{a}x_2, \\ \dot{x}_2=\tilde{b}x_1, \end{cases}$$

where $\tilde{a}$, $\tilde{b}$ is the fuzzy natural number generated by fuzzy number $\tilde{1}$.

II. First method

We consider the following differential equations with fuzzy coefficients

$$\text{(F.D.E.)}\qquad \begin{cases} \dot{x}_1=\tilde{a}x_2, \\ \dot{x}_2=\tilde{b}x_1, \end{cases}$$

where $\tilde{a}$, $\tilde{b}$ is the fuzzy natural number generated by fuzzy number $\tilde{1}$.

We can be expression as follows:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}0 & \tilde{a}\\ \tilde{b} & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\tilde{1}\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

We find the eigenvalues and eigenvectors of $\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}$.

To expand the determinant in the character istic equation

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & a\\ b & -\lambda\end{vmatrix}=0$$

It follows that $\lambda^2-ab=0$Hence the eigenva lues are $\lambda_1=\sqrt{ab}$, $\lambda_2=-\sqrt{ab}$. To find the eigenvectors we must now reduce $(A-\lambda I)K=\tilde{0}$ two times corresponding to the two distinct eigenvalues.

For $\lambda_1=\sqrt{ab}$ we have

$$\begin{pmatrix}-\sqrt{ab} & a\\ b & -\sqrt{ab}\end{pmatrix}\begin{pmatrix}k_1\\ k_2\end{pmatrix}=\begin{pmatrix}\tilde{0}\\ \tilde{0}\end{pmatrix}$$

And solve the following equation:

$$-\sqrt{ab}\,k_1+ak_2=\tilde{0}$$

where zero fuzzy number $\tilde{0}$ satisfies

$$\sqrt{ab}\,k_1-\sqrt{ab}\,k_1=\tilde{0}.$$

Thus we see that $k_2=\frac{\sqrt{ab}}{a}k_1$. Choosing $k_1=\sqrt{ab}\ \tilde{1}$ we get the eigenvector

$$K_1=\tilde{1}\begin{pmatrix}\sqrt{ab}\\ b\end{pmatrix}.$$

For $\lambda_2=-\sqrt{ab}$

$$\begin{pmatrix}\sqrt{ab} & a\\ b & \sqrt{ab}\end{pmatrix}\begin{pmatrix}k_1\\ k_2\end{pmatrix}=\begin{pmatrix}\tilde{0}\\ \tilde{0}\end{pmatrix}.$$

And solve the following equation:

$$\sqrt{ab}\,k_1+a\,k_2=\tilde{0}$$

where zero fuzzy number $\tilde{0}$ satisfies $\sqrt{ab}\,k_1-\sqrt{ab}\,k_1=\tilde{0}$.

Thus we see that $k_2=-\frac{\sqrt{ab}}{a}k_1$

Choosing $k_1=\sqrt{ab}\ \tilde{1}$ then yields the second eigenvector $K_2=\tilde{1}\begin{pmatrix}\sqrt{ab}\\ -b\end{pmatrix}$.

Hence we know that the fuzzy solution of (F.D.E) is given by

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}=c_1K_1e^{\lambda_1 t}+c_2K_2e^{\lambda_2 t}$$

$$= c_1 \tilde{1}\begin{pmatrix}\sqrt{ab}\\ b\end{pmatrix} e^{\sqrt{ab}t} + c_2 \tilde{1}\begin{pmatrix}\sqrt{ab}\\ -b\end{pmatrix} e^{-\sqrt{ab}t}$$

$$= \tilde{1}\left(c_1\begin{pmatrix}\sqrt{ab}\\ b\end{pmatrix} e^{\sqrt{ab}t} + c_2\begin{pmatrix}\sqrt{ab}\\ -b\end{pmatrix} e^{-\sqrt{ab}t}\right).$$

The α-level set of x_1 and x_2 are

$$[x_1]^\alpha = [(c_1\sqrt{ab}e^{\sqrt{ab}t} + c_2\sqrt{ab}e^{-\sqrt{ab}t})\ \tilde{1}_l^\alpha,\ (c_1\sqrt{ab}\, e^{\sqrt{ab}t} + c_2\sqrt{ab}\, e^{-\sqrt{ab}t})\ \tilde{1}_r^\alpha,]$$

$$[x_2]^\alpha = [(c_1 b e^{\sqrt{ab}t} - c_2 b e^{-\sqrt{ab}t})\ \tilde{1}_l^\alpha,\ (c_1 b e^{\sqrt{ab}t} - c_2 b e^{-\sqrt{ab}t})\ \tilde{1}_r^\alpha\ ].$$

III. Second method

We consider the following differential equations with fuzzy coefficients:

$$\text{(F.D.E.)}\qquad \begin{cases} \dot{x}_1 = \tilde{a}x_2, \\ \dot{x}_2 = \tilde{b}x_1, \end{cases}$$

where $\tilde{a}$, $\tilde{b}$ is the fuzzy natural number generated by fuzzy number $\tilde{1}$.

For $\alpha\in[0,1]$, we can be expression as follows

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & [\tilde{a}]^\alpha\\ [\tilde{b}]^\alpha & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

where $[\tilde{a}]^\alpha = [a\cdot \tilde{1}_l^\alpha,\ a\cdot\tilde{1}_r^\alpha]$,

$[\tilde{b}]^\alpha = [b\cdot \tilde{1}_l^\alpha,\ b\cdot\tilde{1}_r^\alpha]$.

First, for each $\alpha\in[0,1]$ we consider the equation

$$\begin{pmatrix}\dot{x}_{1l}{}^\alpha\\ \dot{x}_{2l}{}^\alpha\end{pmatrix} = \begin{pmatrix}0 & a\cdot\tilde{1}_l^\alpha\\ b\cdot\tilde{1}_l^\alpha & 0\end{pmatrix}\begin{pmatrix}x_{1l}^\alpha\\ x_{2l}^\alpha\end{pmatrix}.$$

We find the eigenvalues and eigenvectors of

$$\begin{pmatrix}0 & a\cdot\tilde{1}_l^\alpha\\ b\cdot\tilde{1}_l^\alpha & 0\end{pmatrix}.$$

From the characteristic equation we obtain the real eigenvalues

$$\lambda_1 = \sqrt{ab}\,\tilde{1}_l^\alpha\ ,\ \lambda_2 = -\sqrt{ab}\,\tilde{1}_l^\alpha\ .$$

For $\lambda_1 = \sqrt{ab}\,\tilde{1}_l^\alpha$ we have

$$\begin{pmatrix}-\sqrt{ab}\ \tilde{1}_l^\alpha & a\cdot\ \tilde{1}_l^\alpha\\ b\cdot\ \tilde{1}_l^\alpha & -\sqrt{ab}\ \tilde{1}_l^\alpha\end{pmatrix}\begin{pmatrix}k_1\\ k_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

And solve the following equation:

$$-\sqrt{ab}\ \tilde{1}_l^\alpha k_1 + a\cdot\tilde{1}_l^\alpha k_2 = 0\ .$$

Choosing $k_1 = \sqrt{ab}$ we get the eigenvector

$$K_1 = \begin{pmatrix}\sqrt{ab}\\ b\end{pmatrix}.$$

Similarly, for $\lambda_2 = -\sqrt{ab}\,\tilde{1}_l^\alpha$ yields the second eigenvector

$$K_2 = \begin{pmatrix}\sqrt{ab}\\ -b\end{pmatrix}.$$

Hence the solution is

$$\begin{pmatrix}x_{1l}^\alpha\\ x_{2l}^\alpha\end{pmatrix} = c_1K_1e^{\lambda_1 t} + c_2K_2e^{\lambda_2 t} = c_1\begin{pmatrix}\sqrt{ab}\\ b\end{pmatrix}e^{\sqrt{ab}\ \tilde{1}_l^\alpha t} + c_2\begin{pmatrix}\sqrt{ab}\\ -b\end{pmatrix}e^{-\sqrt{ab}\ \tilde{1}_l^\alpha t}$$

Second, for each $\alpha\in[0,1]$, we consider the equation

$$\begin{pmatrix}\dot{x}_{1r}{}^\alpha\\ \dot{x}_{2r}{}^\alpha\end{pmatrix} = \begin{pmatrix}0 & a\cdot\tilde{1}_r^\alpha\\ b\cdot\tilde{1}_r^\alpha & 0\end{pmatrix}\begin{pmatrix}x_{1r}^\alpha\\ x_{2r}^\alpha\end{pmatrix}.$$

We find the eigenvalues and eigenvectors of

$$\begin{pmatrix}0 & a\cdot\tilde{1}_r^\alpha\\ b\cdot\tilde{1}_r^\alpha & 0\end{pmatrix}.$$

Using the same method, the solution is

$$\begin{pmatrix}x_{1r}^\alpha\\ x_{2r}^\alpha\end{pmatrix} = c_1\begin{pmatrix}\sqrt{ab}\\ b\end{pmatrix}e^{\sqrt{ab}\ \tilde{1}_r^\alpha t} + c_2\begin{pmatrix}\sqrt{ab}\\ -b\end{pmatrix}e^{-\sqrt{ab}\ \tilde{1}_r^\alpha t}.$$

The α-level set of x_1 and x_2 are

$$[x_1]^\alpha = [c_1\sqrt{ab}e^{\sqrt{ab}\ \tilde{1}_l^\alpha t} + c_2\sqrt{ab}e^{-\sqrt{ab}\ \tilde{1}_l^\alpha t},\ c_1\sqrt{ab}e^{\sqrt{ab}\ \tilde{1}_r^\alpha t} + c_2\sqrt{ab}e^{-\sqrt{ab}\ \tilde{1}_r^\alpha t}],$$

$$[x_2]^\alpha = [c_1be^{\sqrt{ab}\ \tilde{1}_l^\alpha t} - c_2be^{-\sqrt{ab}\ \tilde{1}_l^\alpha t},\ c_1be^{\sqrt{ab}\ \tilde{1}_r^\alpha t} - c_2be^{-\sqrt{ab}\ \tilde{1}_r^\alpha t}].$$

IV. Example

Consider the fuzzy solution of the following differential equations with fuzzy coefficients generated by fuzzy number $\tilde{1}$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & \tilde{2}\\ \tilde{3} & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

where $\tilde{1} = [\tilde{1}_l^\alpha, \tilde{1}_r^\alpha] = [\frac{\alpha+1}{2}, \frac{3-\alpha}{2}]$, $\alpha\in[0,1]$.

From the first method, we obtain the α-level set of the solution x_1 and x_2 are

$$[x_1]^\alpha = [(c_1\sqrt{6}\,e^{\sqrt{6}t} + c_2\sqrt{6}\,e^{-\sqrt{6}t})(\frac{\alpha+1}{2}), (c_1\sqrt{6}\,e^{\sqrt{6}t} + c_2\sqrt{6}\,e^{-\sqrt{6}t})(\frac{3-\alpha}{2})],$$

$$[x_2]^\alpha = [(c_1 3e^{\sqrt{6}t} - c_2 3\,e^{-\sqrt{6}t})(\frac{\alpha+1}{2}), (c_1 3\,e^{\sqrt{6}t} - c_2 3\,e^{-\sqrt{6}t})(\frac{3-\alpha}{2})].$$

From the second method, we obtain the α-level set of the solution x_1 and x_2 are

$$[x_1]^\alpha = [(c_1\sqrt{6}\,e^{\sqrt{6}(\frac{\alpha+1}{2})t} + c_2\sqrt{6}\,e^{-\sqrt{6}(\frac{\alpha+1}{2})t}), (c_1\sqrt{6}\,e^{\sqrt{6}(\frac{3-\alpha}{2})t} + c_2\sqrt{6}\,e^{-\sqrt{6}(\frac{3-\alpha}{2})t})],$$

$$[x_2]^\alpha = [(c_1 3\,e^{\sqrt{6}(\frac{\alpha+1}{2})t} - c_2 3\,e^{-\sqrt{6}(\frac{\alpha+1}{2})t}), (c_1 3\,e^{\sqrt{6}(\frac{3-\alpha}{2})t} - c_2 3\,e^{-\sqrt{6}(\frac{3-\alpha}{2})t})].$$

V. References

[1] P. Diamond and P. E. Kloeden, Metric space of Fuzzy sets, World scientific, (1994).

[2] P. Diamond and P. E. Kloeden, Optimization under uncertaintly, Proceedings 3rd. IPMU Congress, B. Bouchon-Meunier and R. R. Yager, Paris, 247-249, (1990).

[3] P. E. Kloeden, Fuzzy dynamical systems, Fuzzy Sets and Systems, 7, 275-296, (1982).

[4] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24, 301-317, (1987).

[5] A. Kaufmann and M. M. Gupta, Introduction to fuzzy arithmetic, Van Nostrand Reinhold, (1991).

[6] Y. C. Kwun, J. R. Kang, S. Y. Kim, The existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition, Journal of Fuzzy Logic and Intelligent Systems, 10, No. 1, 6-11, (2000).

[7] M. Mizumoto and K. Tanaka, Some properties of fuzzy numbers, North-Holland Publishing Company, (1979).

[8] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, 24, 319-330, (1987).