Proceedings of KFIS 2001 Fall Conference, 2001. 12. 1

On Fuzzy Qoutient Spaces

퍼지 몫공간에 관하여

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ABSTRACT

In this paper, we introduce the concept of fuzzy quotient spaces as the new ways and investigate their some properties.

Keywords and phrases : fuzzy quotient space, fuzzy quotient mapping.

1. Introduction and Preliminaries

C.K.Wong, Pu and Liu introduced the concept of fuzzy quotient spaces, respectively and investigated its some properties. But in this paper, we introduce the concept of fuzzy quotient spaces as the new ways and study their some properties.

Now we will list some concepts and results with respect to fuzzy set theory and fuzzy topology needed in the next sections.

Let $I=[0,1]$. For a set X, let I^X be the collection of all the mappings from X into I. Then each member of I^X, $A: X \to I$, is called a *fuzzy set* in X. In particular, ϕ and X can be considered as fuzzy sets in X defined by $\phi(x)=0$ and $X(x)=1$ for each $x \in X$, respectively. Furthermore, $(I^X, \cup, \cap, {}^c)$ is a completely distributive lattice for which De Morgan's laws hold(cf.[1,5,12]).

The concept of a fuzzy point in a set its related notions and their properties refer to [5,7,9,11]. We will denote the set of all fuzzy points in a set X as $F_p(X)$.

Definition 1.1[1]. Let $f: X \to Y$ be a mapping, $A \in I^X$ and $B \in I^Y$. Then :

(1) The *inverse image of* B *under* f, denoted by $f^{-1}(B)$ is a fuzzy set in X defined by for each $x \in X$, $[f^{-1}(B)](x) = B(f(x)) = (B \circ f)(x)$.

(2) The *image of* A *under* f, denoted by $f(A)$ is a fuzzy set in Y denoted by for each $y \in Y$,

$$f(A) = \begin{cases} sup_{y=f(x)} A(x), & \text{if } y \in f(X) \\ 0, & \text{if } y \notin f(X) \end{cases}$$

By the above definition, $f: I^X \to I^Y$ and $f^{-1}: I^Y \to I^X$ are mappings.

Result 1.A[1,11]. Let $f: X \to Y$, $\{A_\alpha\}_{\alpha \in \Lambda} \subset I^X$, and $\{B_\alpha\}_{\alpha \in \Lambda} \subset I^Y$. Then :

(1) $f^{-1}(\bigcup_{\alpha \in \Lambda} B_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$,
$f^{-1}(\bigcap_{\alpha \in \Lambda} B_\alpha) = \bigcap_{\alpha \in \Lambda} f^{-1}(B_\alpha)$.

(2) $f(\bigcup_{\alpha \in \Lambda} A_\alpha) = \bigcup_{\alpha \in \Lambda} f(A_\alpha)$,
$f(\bigcap_{\alpha \in \Lambda} A_\alpha) = \bigcap_{\alpha \in \Lambda} f(A_\alpha)$.

(3) $f(f^{-1}(B)) \subset B$, for each $B \in I^Y$.
In particular, if f is surjective, then $f(f^{-1}(B)) = B$.

(4) $A \subset f^{-1}(f(A))$, for each $A \in I^X$.
In particular, if f is injective, then $f^{-1}(f(A)) = A$.

(5) Let $g: Y \to Z$ be a mapping. If $B \in I^Z$, then $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$;
If $A \in I^X$, then $(g \circ f)(A) = f(g(A))$.

(6) If f is bijective, then $[f(A)]^c = f(A^c)$ for each $A \in I^X$.

Definition 1.2[1]. A family $\Im$ of fuzzy sets in a set X is called a *fuzzy topology* on X if it satisfies the following conditions:

(a) $\phi, X \in \Im$.
(b) If A, $B \in \Im$, then $A \cap B \in \Im$.
(c) If $\{A_\alpha\}_{\alpha \in \Lambda} \subset \Im$, then $\bigcup_{\alpha \in \Lambda} A_\alpha \in \Im$.

The pair $(X, \Im)$ is called a *fuzzy topological space*(in short, *fts*). Every member of $\Im$ is called a $\Im$*-open*

*supported in part by the grant of Woosuk University in 2001.

fuzzy set(in short, *F-open set*) in X. A fuzzy set A is $\mathfrak{T}$*-closed* if and only if A^c is *F-open* in X.

Notation 1.3. (1) For a fts X, $FO(X)$ and $FC(X)$ denote the family of all F-open sets and F-closed sets in X, respectively.
(2) For a set X, $\mathfrak{T}_F(X)$ denotes the family of all fuzzy topology on X.
It is clear that $(\mathfrak{T}_F(X), \subset)$ is a complete lattice.

Definition 1.4[5]. Let X be an fts, let $A \in I^X$ and let $x_\lambda \in F_p(X)$. Then A is called a:
(1) *fuzzy neighborhood*(in short, *F-nbd*) of x_λ if there exists a $U \in FO(X)$ such that $x_\lambda \in U \subset A$. The family consisting of all the F-nbds of x_λ is called the *system of F-nbds of* x_λ, and will be denoted by $\mathcal{N}_F(x_\lambda)$.
(2) *Q-neighborhood*(in short, *Q-nbd*) of x_λ if there exists a $U \in FO(X)$ such that $x_\lambda q\, U \subset A$. The family consisting of all the Q-nbds of x_λ is called the *system of Q-nbds of* x_λ, and will be denoted by $\mathcal{N}_Q(x_\lambda)$.

Definition 1.5[4,5]. Let X be a fts and let $A \in I^X$. Then the *closure* of A, denoted by $cl\,A$ is defined by: $cl\,A = \bigcap\{F \in FC(X) : A \subset F\}$.
It is clear that $cl\,A$ is the smallest F-closed set containing A and $cl(cl\,A) = cl\,A$.

Definition 1.6[4,5]. Let X be a fts and let $A \in I^X$. Then the union of all the F-open sets contained in A is called the *interior* of A and denoted by $int\,A$. Hence $int\,A = \bigcup\{U \in FO(X) : U \subset A\}$.
It is clear that $int\,A$ is the largest F-open set contained in A and $int(int\,A) = int\,A$.

Definition 1.7[1]. A mapping $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$ is said to be *fuzzy continuous*(in short, *F-continuous*) if $f^{-1}(B) \in \mathfrak{T}$ for each $B \in \mathfrak{T}^*$. The mapping f is called a *fuzzy homeomorphism*(in short, *F-homeomorphism*) if f is bijective, and both f and f^{-1} are F-continuous.

Result 1.B[1]. If $f : X \to Y$ and $g : Y \to Z$ are F-continuous, then $g \circ f : X \to Z$ is F-continuous.

Definition 1.8[1,8]. Let X and Y be fts's. Then a mapping $f : X \to Y$ said to be :
(1) *fuzzy open*(in short, *F-open*) if for each $U \in FO(X)$, $f(U) \in FO(Y)$.
(2) *fuzzy closed*(in short, *F-closed*) if for each $F \in FC(X)$, $f(F) \in FC(Y)$.

Definition 1.9[6,8]. Let $(X, \mathfrak{T}_X)$ be fts, let R an equivalence relation on X, let X/R the usual quotient set and let $\pi : X \to X/R$ the usual projection(quotient mapping). Then the collection $\mathfrak{T}_{X/R} = \{B \in I^{X/R} : \pi^{-1}(B) \in \mathfrak{T}_X\}$ is a fuzzy topology on X/R. In this case, $\mathfrak{T}_{X/R}$ is called the *fuzzy quotient topology on* X/R, the pair $(X/R, \mathfrak{T}_{X/R})$ the *fuzzy quotient space* of $(X, \mathfrak{T})$ and π the *fuzzy quotient mapping*.

Result 1.C[8, Theorem 4.1]. (1) $\mathfrak{T}_{X/R}$ is the largest fuzzy topology on X/R for which π is F-continuous.
(2) Let $(Y, \mathfrak{T}_Y)$ be a fts and let $g : (X/R, \mathfrak{T}_{X/R}) \to (Y, \mathfrak{T}_Y)$ a mapping. Then g is F-continuous if and only if $g \circ \pi$ is F-continuous.

2. Fuzzy quotient spaces defined by the first way

After we consider the following result, we introduce the concept of a fuzzy quotient space defined by the first way and study its some properties.

Result 2.A[2]. Let $(X, \mathfrak{T}_X)$ be a fts, let Y a set and let $f: X \to Y$ a mapping. Let $\mathfrak{T}_Y = \{U \in I^Y : f^{-1}(U) \in \mathfrak{T}_X\}$. Then we have the following properties :
(a) $\mathfrak{T}_Y$ is a fuzzy topology on Y.
(b) $f: X \to Y$ is F-continuous.
(c) If U is a fuzzy topology on Y such that $f: X \to (Y, U)$ is F-continuous, then $\mathfrak{T}_Y$ is finer than U.

Definition 2.1. Let $(X, \mathfrak{T}_X)$ be a fts, let Y a set, and let $f: X \to Y$ a surjection. Then $\mathfrak{T}_Y = \{U \in I^Y : f^{-1}(U) \in \mathfrak{T}\}$ is called the *fuzzy quotient topology on* Y *induced by* f. The pair $(Y, \mathfrak{T}_Y)$ is called a *fuzzy quotient space of* X and f a *fuzzy quotient mapping*.

By Result 2.A, the fuzzy quotient mapping f is not only F-continuous, but $\mathfrak{T}_Y$ is the finest fuzzy topology on Y for which f is F-continuous.

The following result is an immediate consequence of Definition 2.1 :

Proposition 2.2. Let $(Y, \mathfrak{T}_Y)$ be a fuzzy quotient space of a fts $(X, \mathfrak{T}_X)$ with fuzzy quotient mapping f. Then $F \in FC(Y)$ if and only if

$f^{-1}(F) \in FC(X)$.

Theorem 2.3. Let $(X, \mathfrak{I}_X)$ and $(Y, \mathfrak{I})$ be fts's, let $f: X \to Y$ F-continuous and surjective, and let $\mathfrak{I}_Y$ the fuzzy quotient topology on Y induced by f. If f is F-open or F-closed, then $\mathfrak{I}_Y = \mathfrak{I}$.

(Proof) Suppose f is F-open. By Result 2.A(c), $\mathfrak{I} \subset \mathfrak{I}_Y$. Let $U \in \mathfrak{I}_Y$. Then $f^{-1}(U) \in \mathfrak{I}_X$, by the definition of $\mathfrak{I}_Y$. Since f is F-open, $f(f^{-1}(U)) \in \mathfrak{I}$. Since f is surjective, $f(f^{-1}(U)) = U$. Thus $U \in \mathfrak{I}$. So $\mathfrak{I}_Y \subset \mathfrak{I}$. Hence $\mathfrak{I}_Y = \mathfrak{I}$.

Now suppose f is F-closed. It is sufficent to show that $\mathfrak{I}_Y \subset \mathfrak{I}$. Let $U \in \mathfrak{I}_Y$. Then $f^{-1}(U) \in \mathfrak{I}_X$ by the Definition 2.1 of $\mathfrak{I}_Y$. Thus $[f^{-1}(U)]^c = f^{-1}(U^c) \in FC(X)$. Since f is F-closed, $f(f^{-1}(U^c))$ is F-closed in $(Y, \mathfrak{I}_Y)$. Since f is surjective, $f(f^{-1}(U^c)) = U^c$. Thus U^c is F-clsed in $(Y, \mathfrak{I}_Y)$. So $U \in T$ and thus $\mathfrak{I}_Y \subset \mathfrak{I}$. Hence $\mathfrak{I}_Y = \mathfrak{I}$.

Theorem 2.3 tells us that if $f: (X, \mathfrak{I}_X) \to (Y, \mathfrak{I})$ is F-open(or F-closed), F-continuous and furjective, then f is a fuzzy quotient mapping.

Theorem 2.4. The composition of two fuzzy quotient mappings is a fuzzy quotient mapping.

Theorem 2.5. Let $(X, \mathfrak{I}_X)$ be a fts, let Y a set, let $f: X \to Y$ surjective, let $\mathfrak{I}_Y$ the fuzzy quotient topology on Y induced by f and let $(Z, \mathfrak{I}_Z)$ a fts. Then a mapping $g: Y \to Z$ is F-continuous if and only if $g \circ f: X \to Z$ is F-continuous.

Theorem 2.6. Let $(X, \mathfrak{I}_X)$ and $(Y, \mathfrak{I}_Y)$ be fts's and let $p: X \to Y$ F-continuous and surjective. Then p is a fuzzy quotient mapping if and only if for each fts $(Z, \mathfrak{I}_Z)$ and each mapping $g: Y \to Z$, the F-continuity of $g \circ p$ implies that of g.

Theorem 2.7. Let $(X, \mathfrak{I}_X)$, $(Y, \mathfrak{I}_Y)$ and $(Z, \mathfrak{I}_Z)$ be fts's, let $p: X \to Y$ a fuzzy quotient mapping and let $h: X \to Z$ F-continuous. Suppose $h \circ p^{-1}$ is single-valued ; i.e., for each $y \in Y$, h is constant on $p^{-1}(y)$. Then :

(a) $(h \circ p^{-1}) \circ p = h$ and $h \circ p^{-1}$ is F-continuous.

(b) $h \circ p^{-1}$ is an F-open(F-closed) mapping if and only if $h(U)$ is F-open(F-closed) in Z whenever U is F-open(F-closed) in X satisfying $U = p^{-1}(p(U))$.

Theorem 2.8. Let $(X, \mathfrak{I}_X)$, $(Y, \mathfrak{I}_Y)$ and $(Z, \mathfrak{I}_Z)$ be fts, let $p: X \to Y$ a fuzzy quotient mapping and let $g: Y \to Z$ a surjection. Then $g \circ p$ is a fuzzy quotient mapping if and only if g is a fuzzy quotient mapping.

Lemma 2.9. Let $f: X \to Y$ be an F-continuous mapping, let $\sim_f$ the equivalence relation on X induced by f and let $\pi: X \to X/\sim_f$ the natural mapping. Let $(X/\sim_f, T_{X/\sim})$ be a fuzzy quotient space of X. Then $f \circ \pi^{-1}$ is F-continuous and injective.

Furthermore, if f is surjective, then $f \circ \pi^{-1}$ is bijective.

Theorem 2.10. Let $(X, \mathfrak{I}_X)$ and $(Y, \mathfrak{I}_Y)$ be fts's and let $f: X \to Y$ be F-continuous and surjective. Then $f \circ \pi^{-1}: X/_{\sim f} \to Y$ is an F-homeomorphism if and only if f is a fuzzy quotient mapping.

3. Fuzzy quotient spaces defined by the second way

We turn our attention toward the second way of defining a fuzzy quotient space.

Definition 3.1. Let $(X, \mathfrak{I}_X)$ be a fts and let $\mathscr{B}$ a partition of X. Define a mapping $p: X \to \mathscr{B}$ as follows : For each $x \in X$, $p(x) \in \mathscr{B}$ contains x. Let $\mathfrak{I}_{\mathscr{B}}$ be the fuzzy quotient topology on $\mathscr{B}$ induced by p. Then $(\mathscr{B}, \mathfrak{I}_{\mathscr{B}})$ is called a *fuzzy quotient space* of X. The mapping p is called the *natural mapping* of X onto $\mathscr{B}$. The set is also called a *decomposition* of X and the fuzzy quotient space $(\mathscr{B}, \mathfrak{I}_{\mathscr{B}})$ is also called a *fuzzy decomposition space* or a *fuzzy identification space* of X.

Example 3.2. Let $X = \{a, b, c\}$, let $\mathscr{B} = \{\{a, b\}, \{c\}\}$, and let $\mathfrak{I}_X = \{\emptyset, X, O_1, O_2, O_1 \cap O_2, O_1 \cup O_2\}$, where $O_1 = \{(a, 0.3), (b, 0.3), (c, 0.8)\}$ and $O_2 = \{(a, 0.6), (b, 0.6), (c, 0.7)\}$.

Let the natural mapping $p: X \to \mathscr{B}$ be defined by :

$p(a) = p(b) = \{a, b\}$ and $p(c) = \{c\}$.

Then the fuzzy quotient topology $\mathfrak{I}_{\mathscr{B}}$ on $\mathscr{B}$ induced by p is as follows :

$\mathfrak{I}_{\mathscr{B}} = \{\emptyset, X, O_1^*, O_2^*, O_1^* \cap O_2^*, O_1^* \cup O_2^*\}$,

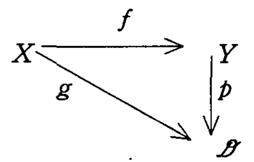
where $O_1^* = \{(\{a, b\}, 0.3), (\{c\}, 0.8)\}$ and $O_2^* = \{(\{a, b\}, 0.6), (\{c\}, 0.7)\}$.

Lemma 3.3. let X be a set, let $\mathscr{D}$ a partition of X and let $p: X \to \mathscr{D}$ the natural mapping. Then for each $\xi \in I^{\mathscr{D}}$, $p^{-1}(\xi) = \bigcup \xi$.

Theorem 3.4. Let $(X, \Im_X)$ be a fts, let a decomposition of X, let $p: X \to \mathscr{D}$ the natural mapping, let $\Im_{\mathscr{D}}$ the fuzzy quotient topology on $\mathscr{D}$ and let $\xi \in I^{\mathscr{D}}$. Then $\xi \in \Im_{\mathscr{D}}$ if and only if $\bigcup \xi \in \Im_X$.

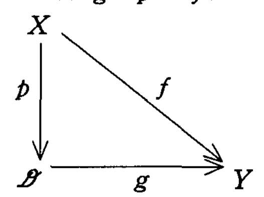
Theorem 3.5. let $(X, \Im_X)$ be a fts, let Y a set, let $f: X \to Y$ a surjection, and let $\Im_Y$ the fuzzy qoutient topology on Y induced by f. Let $D = \{f^{-1}(y): y \in Y\}$. Then there exists a F-homeomorhism $g: Y \to \mathscr{D}$ such that $g \circ f$ is the natural mapping $p: X \to \mathscr{D}$.

Theorem 3.5 yields the following commutative diagram, where g is a F-homeomorphism :



The following theorem provides a criterion for determining when a mapping whoce demain is a fuzzy quotient space is F-continuous.

Theorem 3.6. Let $(X, \Im_X)$ be a fts, let $(\mathscr{D}, \Im_{\mathscr{D}})$ be a fuzzy decomposition space of X, and let $p: X \to \mathscr{D}$ the natural mapping. Let $(Y, \Im_Y)$ be a fts and let $f: X \to Y$ a F-continuous mapping such that for each $D \in \mathscr{D}$, $f(x) = f(y)$ for all $x, y \in D$. Then there exists a F-continuous mapping $g: \mathscr{D} \to Y$ such that $g \circ p = f$.



Theorem 3.7. Let $(X, \Im_X)$ and $(Y, \Im_Y)$ be fts's, let $f: X \to Y$ F-continuous surjective, let $D = \{f^{-1}(y): y \in Y\}$, and let $\Im_{\mathscr{D}}$ the fuzzy quotient topology on $\mathscr{D}$ induced by the natural mapping $p: X \to \mathscr{D}$. Then the induced F-continuous mapping(given in Theorem 3.6) $g: \mathscr{D} \to Y$ is a F-homeomorphism if and only if f is a fuzzy quotient mapping.

The following theorem gives a criterion for a fuzzy quotient space to be a fuzzy hausdorff space in the sense of Pu and Liu.

Theorem 3.8. Let $(X, \Im_X)$ and $(Y, \Im_Y)$ be fts, let $f: X \to Y$ F-continu- ous surjective, let $D = \{f^{-1}(y): y \in Y\}$, and let $\Im_D$ the fuzzy quotient topology on $\mathscr{D}$ induced by the natural mapping $p: X \to \mathscr{D}$. If Y is FT_2 in the sense of Pu and Liu, then so is $\mathscr{D}$.

Remark 3.9. Defintion 1.9, is the third way of defining a fuzzy quotient space.

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