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t-norm의 크기에 대한 고찰

A note on a triangular norm hierarchy

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ABSTRACT

In Cretu [Fuzzy Sets and Systems 120(2001) 371-383], triangular norms and their hierarchy are investigated. In this paper, we give new proofs which are significantly shorter than those given in Cretu, applying a known result which involves only one argument of one-place rather than two place arguments by Klement et al.[FSS 86(1997) 189-195]

Keywords : *t*-norm; Fuzzy logic; Comparison of *t*-norms

1. Introduction

Triangular norms *t*-norms and the corresponding *t*-conorms are used in several branches of mathematics in different manners, e.g., in probabilistic metric spaces, many-valued logic, fuzzy sets, decomposable measures and their applications [2, 5, 8, 10, 13]. A *t*-norm T is a two-place function from the unit square into the unit interval which is associative, commutative, non-decreasing, and fulfills, for all x in[0, 1], the boundary condition $T(1,x)=x$.
Its dual function S defined via $S(x,y)=1-T(1-x,1-y)$ is called a t-conorm (see[11]).

We are now interested in the question whether, given two *t*-norms T_1 and T_2, T_1 is weaker than T_2 or, equivalently, T_2 is stronger than T_1 (in symbols $T_1\leq$ and T_2, i.e., $T_1(x,y)\leq$ and $T_2(x,y)$ for all points (x,y) in the unit square.

Cretu[3], recently, showed the monotonicity of some well-known classes of *t*-norms. In this paper, We simplify this result, applying a known result which involves only one argument of one-place rather than two place arguments by Klement et al.[FSS 86(1997) 189-195]

2. Some known results about t-norm

The following are the most important *t*-norms, together with their corresponding *t*-conorms:

$T_M(x,y)=\min(x,y)$, $S_M(x,y)=\max(x,y)$,

$T_P(x,y)=x\cdot y$, $S_P(x,y)=x+y-x\cdot y$,

$T_L(x,y) = \max(x+y-1, 0)$,

$S_L(x,y) = \min(x+y, 1)$,

The following lemma is obvious from the monotonicity and boundary conditions.

Lemma 2.1. Let T be a t-norm. Then the following statement holds

$$T(x,y) \leq T_M(x,y), \forall x, y \in [0,1].$$

Continuous t-norms (t-conorms) were studied extensively by Ling [8], among others. A continuous t-norm T is called Archimedean if $T(x,y) < x$ for all $x \in (o,1)$. A continuous t-norm T is strict if $T(x,y) < T(x,y)$ whenever $x \in (0,1)$ and $y < z$. Each strict t-norm T is Archimedean. Non-strict continuous Archimedean t-norms are called nilpotent. Aczel[1], Mostert and Shields [9] and Ling [7] have proved the following result:

Theorem 2.1. T is a continuous Archimedean t-norm if and only if there is a continuous strictly decreasing function:

$f: [0,1] \to [0, \infty]$ such that $f(1) = 0$ and

$T(a,b) = f^*(f(a) + f(b))$,

where f^* is the pseudoinverse of f, i.e., for all $x \in [0, \infty]$,

$$f^*(x) = f^{-1}(\min(x, f(0)))$$

T is strict if and only if $f(0) = +\infty$, i.e., f is bijective and $f^* = f^{-1}$.

The function f is called an additive generator of T and it is unique up to a positive multiplicative constant.

Now, let T_1, T_2 be two continuous Archimedean t-norms with additive generators f_1 and f_2, respectively, The full information about T_1 is contained in f_i and, as a consequence, it should be possible to decide whether T_1 is weaker than T_2 only by means of f_1 and f_2.

The first step into this direction was done by Schweizer and Sklar [11, Theorem 7], who proved that if both T_1 and T_2 are strict, then $T_1 \leq T_2$ if and only if the composite $h = f_1 \circ f_2^{-1}$ is a subadditive function, i.e., if for all $s, t \geq 0$

$$h(s+t) \leq h(s) + h(t)$$

Klement et al. [6, Corollary 3.2] showed the following result as a corollary of this result

Theorem 2.2[6]. Let T_1, T_2 be two continuous Archimedean t-norms with differentiable additive generators f_1 and f_2, respectively. If $g = f_1' / f_2'$ is a non-decreasing function on (0,1), then we have $T_1 \leq T_2$.

3. t-norm hierarchy

In this section, we reconsider families of t-norms which are investigated by Cretu[3]. The reasonings are significantly shorten the the proofs given in [6]. Many applications deal with the Frank[4] family of t-norms, where for $s \in [0, \infty]$

$$T_s^F(x,y) = \begin{cases} T_M(x,y) & \text{if } s=0, \\ T_P(x,y) & \text{if } s=1, \\ T_L(x,y) & \text{if } s=\infty, \\ \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right) & \text{otherwise} \end{cases}$$

Cretu[3] showed at Proposition 2.1, 2.2, and 2.3 in his paper that,

for $0 < r < 1 < s < \infty$,

$$T_\infty^F \leq T_s^F \leq T_1^F \leq T_r^F$$

But Klement et al.[6] already gave a proof using Theorem 2.2 which is significantly shorter than that given in [2,3]. Here we summarize them.

Frank showed that this family is continuous with respect to the parameter s. Note that trivially $T_0^F = T_M \geq T_s^F$ for all $s \in (0,\infty)$. For each $s \in (0,\infty)$, T_s^F is a strict t-norm whose generator is given by

$$f_s(x) = \begin{cases} -\log x & \text{if } s=1, \\ \log \frac{s-1}{s^x-1} & \text{if } s \neq 1. \end{cases}$$

T_∞^F is a nilpotent t-norm, its generator is given by $f_\infty(x) = 1-x$. Then

$$\begin{cases} \frac{f_\infty(\mu)}{f_s(\mu)} = \frac{1}{\log s}(1-s^{-\mu}) & \text{for } s \in (0,\infty)\setminus\{1\}, \\ \frac{f_\infty(\mu)}{f_1(\mu)} = \mu & \text{for } s \in (0,\infty), \\ \frac{f_t(\mu)}{f_s(\mu)} = \frac{\log s}{\log t}\,\frac{1-b^\mu}{1-a^\mu} & \text{for } 1<s<t<\infty, \\ & \text{(the case } 0<s<t<1 \text{ is completely analogous)} \end{cases}$$

are non-decreasing on $(0,1)$. Hence, Theorem 2.2 implies the following results which are proposition 2.1, 2.2, and 2.3 in [3].

Proposition 1. Let $s \in (0,1)$, $T_s(x,y) = \log_s(1+(s^x-1)(s^y-1)/(s-1))$ and $T_1(x,y) = xy$, $\forall x,y \in [0,1]$.
Then

$$T_s(x,y) \geq T_1(x,y),\ \forall x,y \in [0,1].$$

Proposition 2. Let $s \in (1,\infty)$, $T_s(x,y) = \log_s(1+(s^x-1)(s^y-1)/(s-1))$ and $T_1(x,y) = xy$, $\forall x,y \in [0,1]$.
Then

$$T_s(x,y) \leq T_1(x,y),\ \forall x,y \in [0,1].$$

Proposition 3. Let $s \in (1,\infty)$, $T_s(x,y) = \log_s(1+(s^x-1)(s^y-1)/(s-1))$ and $T_\infty(x,y) \leq T_s(x,y)$, $\forall x,y \in [0,1]$.
Then

$$T_\infty(x,y) \leq T_s(x,y),\ \forall x,y \in [0,1].$$

The following two propositions, which are proposition 2.4 and 2.5 in [3], are trivial by Lemma 2.1.

Proposition 4. Let $s \in (0,1)$, $T_s(x,y) = \log_s(1+(s^x-1)(s^y-1)/(s-1))$ and $T_0(x,y) = \min(x,y)$, $\forall x,y \in [0,1]$.
Then

$$T_s(x,y) \leq T_0(x,y),\ \forall x,y \in [0,1].$$

Proposition 5. Let $a \in [0,1]$ and $T^a(x,y) = \frac{xy}{\max(x,y,a)}$ (Dubois and Prade intersection), $T_0(x,y) = \min(x,y)$.
Then

$$T^a(x,y) \leq T_0(x,y),\ \forall x,y \in [0,1].$$

Next, we give a simple proof of proposition 2.12[3].

Proposition 6. Let $T_{Yager(p)}: [0,1]\times[0,1] \to [0,1]$

$T_{Yager(p)}(x,y) = 1-\min(1, ((1-x)^p + (1-y)^p)^{\frac{1}{p}}$, $p \geq 1$.
Then

$$T_\infty(x,y) \leq T_{Yager(p)}(x,y) \leq T_0(x,y).$$

Finally, we prove the following two propositions which are Proposition 3.1 and 3.2 in [3].

Proposition 7. Let $T_{Einstein}: [0,1]\times[0,1] \to [0,1]$

$$T_{Einstein}(x,y) = \frac{xy}{1+(1-x)(1-y)}.$$

Then

$$T_\infty(x,y) \leq T_{Einstein}(x,y) \leq T_1(x,y),\ \forall x,y \in [0,1].$$

Proposition 8. Let $T_{Hamacher(\lambda)}: [0,1]\times[0,1] \to [0,1]$

$T_{Hamacher(\lambda)}(x,y) = \frac{xy}{\lambda+(1-\lambda)(x+y-xy)}$, $\lambda \in [1,2]$
Then

$$T_\infty(x,y) \leq T_{Hamacher(\lambda)}(x,y) \leq T_1(x,y).$$

Since $T_{Hamacher(2)} = T_{Einstein}$ it is enough to prove Proposition 8.

4. Conclusion

We reconsidered most results about t-norm hierarchy given by Cretu[3] and gave significantly simple proofs using a known result which involves only one argument of one-place rather than two place arguments by Klement et al.[FSS 86(1997) 189-195].

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