Several decompositions of fuzzy transformation semigroups

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Abstract
We introduce sums and joins of fuzzy finite state machines and investigate their algebraic structures.

Key Words : Fuzzy finite state machine, covering, cascade product, wreach product, sums and joins.

1. Introduction

Since Wee [9] in 1967 introduced the concept of fuzzy automata following Zadeh [10], fuzzy automata theory has been developed by many researchers. Recently Malik et al.[5-8] introduced the concepts of fuzzy finite state machines and fuzzy transformation semigroups based on Wee's concept [9] of fuzzy automata and related concepts and applied algebraic technique. Cho et al. [2, 4] introduced the notion of a T-fuzzy finite state machine that is an extension of a fuzzy finite state machine. Even if $T = \wedge$, our notion is different from the notion of Malik et al. [6, 7].

In this paper, we introduce sums and joins of fuzzy finite state machines that are generalizations of crisp concepts in algebraic automata theory and investigate their algebraic structures. For the terminology in (crisp) algebraic automata theory, we refer to [3].

2. Preliminaries

Definition 2.1[1,4]. A triple $M = (Q, X, \tau)$ where Q and X are finite nonempty sets and τ is a fuzzy subset of $Q \times X \times Q$, i.e., τ is a function from $Q \times X \times Q$ to [0, 1], is called a fuzzy finite state machine if $\sum_{q \in Q} \tau(p, a, q) \leq 1$ for all $p \in Q$ and $a \in X$.

Definition 2.2 [4]. Let $M_1 = (Q_1, X_1, \tau_1)$ and $M_2 = (Q_2, X_2, \tau_2)$ be fuzzy finite state machines. Let $\alpha : Q_1 \rightarrow Q_2$ and $\beta : X_1 \rightarrow X_2$ be mappings. Then the pair (α, β) is called a fuzzy finite state machine homomorphism (which is written by (α, β)) if

$$\tau_1(p, a, q) \leq \tau_2(\alpha(p), \beta(a), \alpha(q)),$$

$$p, q \in Q_1, a \in X_1.$$

The homomorphism $(\alpha, \beta) : M_1 \to M_2$ is called isomorphism if α and β are bijective respectively.

Definition 2.3 [1,4]. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. If $\xi : X_1 \to X_2$ is a function and $\eta : Q_2 \to Q_1$ is a surjective partial function such that

$$\tau_1(\eta(p), a, \eta(q)) \leq \tau_2(p, \xi(a), q)$$

for all p, q in the domain of η and $a \in X_1$, then we say that (η, ξ) is a covering of M_1 by M_2 and that M_2 covers M_1 and denote by $M_1 \leq M_2$. Moreover, if the inequality always turns out equality, then we say that (η, ξ) is a complete covering of M_1 by M_2 and that M_2 completely covers M_1 and denote by $M_1 \leq_c M_2$.

Proposition 2.4. Let M_1, M_2 and M_3 be fuzzy finite state machines. If $M_1 \leq M_2$ [resp. $M_1 \leq_c M_2$] and $M_2 \leq M_3$ [resp. $M_2 \leq_c M_3$], then $M_1 \leq M_3$ [resp. $M_1 \leq_c M_3$].

3. Several products of fuzzy finite state machines

Several products of finite state machines are in [3]. Some of these products have been fuzzified in [1], [4] and [6]. In this section we introduce sums and joins of fuzzy finite state machines.

Definition 3.1 [1,6]. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. The cascade product $M_1 \omega M_2$ of M_1 and M_2 with respect to $\omega : Q_2 \times X_2 \to X_1$ is the fuzzy finite state machine $(Q_1 \times Q_2, X_2, \tau_1 \omega \tau_2)$ with

$$(\tau_1 \omega \tau_2)((p_1, p_2), b, (q_1, q_2))$$

$$= \wedge(\tau_1(p_1, \omega(p_2, b), q_1), \tau_2(p_2, b, q_2)).$$

Definition 3.2 [1,6]. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. The wreath product $M_1 \circ M_2$ of M_1 and M_2 is the fuzzy finite state machine

$(Q_1 \times Q_2, X_1^{Q_2} \times X_2, \tau_1 \circ \tau_2)$ with

$$(\tau_1 \circ \tau_2)((p_1, p_2), (f, b), (q_1, q_2))$$

$$= \wedge(\tau_1(p_1, f(p_2), q_1), \tau_2(p_2, b, q_2)).$$

Now we introduce sums and joins of fuzzy finite state machines.

Definition 3.3. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines, where $Q_1 \cap Q_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. The join $M_1 \vee M_2$ of M_1 and M_2 is the fuzzy finite state machine

$(Q_1 \cup Q_2, X_1 \cup X_2, \tau_1 \vee \tau_2)$ with

$$(\tau_1 \vee \tau_2)(p, a, q) =$$

$$\begin{cases} \tau_1(p, a, q) \text{ if } (p, a, q) \in Q_1 \times X_1 \times Q_1 \\ \tau_2(p, a, q) \text{ if } (p, a, q) \in Q_2 \times X_2 \times Q_2 \\ 0, \textit{ otherwise} \end{cases}$$

Definition 3.4. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines, where $Q_1 \cap Q_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. The join* $M_1 \vee^* M_2$ of M_1 and M_2 is the fuzzy finite state machine

$(Q_1 \cup Q_2, X_1 \cup X_2, \tau_1 \vee^* \tau_2)$ with

$$(\tau_1 \vee^* \tau_2)(p, a, q) = \begin{cases} \tau_1(p, a, q) \text{ if } (p, a, q) \in Q_1 \times X_1 \times Q_1 \\ \tau_2(p, a, q) \text{ if } (p, a, q) \in Q_2 \times X_2 \times Q_2 \\ 1 \text{ if } (p, a, q) \in (Q_1 \times X_1 \times Q_2) \cup (Q_2 \times X_2 \times Q_1) \\ 0, \textit{ otherwise} \end{cases}$$

Definition 3.5. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines, where $Q_1 \cap Q_2 = \emptyset$. The sum $M_1 + M_2$ of M_1 and M_2 is the fuzzy finite state machine

$(Q_1 \cup Q_2, X_1 \times X_2, \tau_1 + \tau_2)$ with

$$(\tau_1 + \tau_2)(p, (a, b), q) = \begin{cases} \tau_1(p, a, q) \text{ if } p, q \in Q_1 \\ \tau_2(p, b, q) \text{ if } p, q \in Q_2 \\ 0, \textit{ otherwise} \end{cases}$$

4. Associative properties

Proposition 4.1 [4]. Let M_1, M_2 and M_3 be fuzzy finite state machines. Then the following are hold:

(i) $(M_1 \wedge M_2) \wedge M_3 = M_1 \wedge (M_2 \wedge M_3)$.

(ii) $(M_1 \times M_2) \times M_3 = M_1 \times (M_2 \times M_3)$.

Now we prove that wreath product, join and sum of fuzzy finite state machines are associative.

Theorem 4.2 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ and $M_3=(Q_3, X_3, \tau_3)$ be fuzzy finite state machines. Then the following hold:

(i) $(M_1 \circ M_2) \circ M_3 \cong M_1 \circ (M_2 \circ M_3)$

(ii) $(M_1 \vee M_2) \vee M_3 \cong M_1 \vee (M_2 \vee M_3)$, where $Q_1 \cap Q_2 \cap Q_3 = \emptyset$ and $X_1 \cap X_2 \cap X_3 = \emptyset$

(iii) $(M_1 + M_2) + M_3 \cong M_1 + (M_2 + M_3)$, where $Q_1 \cap Q_2 \cap Q_3 = \emptyset$

Remark. $\vee^*$ is not an associative operation.

5. Coverings

Proposition 5.1. Let M_1 and M_2 be fuzzy finite state machines. Then

(i) $M_1 \wedge M_2 \leq_c M_1 \times M_2$

where $X_1 = X_2$

(ii) $M_1 \omega M_2 \leq_c M_1 \circ M_2$

Proposition 5.2. Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines such that $Q_1 \cap Q_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. Then

(i) $M_1 \leq M_1 \vee M_2$

(ii) $M_1 \leq M_1 \vee^* M_2$

Theorem 5.3. Let M_1 and M_2 be fuzzy finite state machines. Then

(i) $M_1 \vee M_2 \leq M_1 \vee^* M_2$

(ii) $M_1 + M_2 \leq M_1 +^* M_2$

Theorem 5.4. Let M_1, M_2 and M_3 be fuzzy finite state machines such that $M_1 \leq M_2$. Then

(i) $M_1 \vee M_3 \leq M_2 \vee M_3$

(ii) $M_1 \vee^* M_3 \leq M_2 \vee^* M_3$

(iii) $M_1 + M_3 \leq M_2 + M_3$

Theorem 5.5. Let M_1, M_2 and M_3 be fuzzy finite state machines such that $M_1 \leq M_2$. Then

(i) $M_1 \circ M_3 \leq M_2 \circ M_3$

(ii) $M_3 \circ M_1 \leq M_3 \circ M_2$

Theorem 5.6. Let $M_1 = (Q_1, X_1, \tau_1)$ and $M_2 = (Q_2, X_2, \tau_2)$ and $M_3 = (Q_3, X_3, \tau_3)$ be fuzzy finite state machines such that $Q_2 \cap Q_3 = \emptyset$. Then

(i) $M_1 \circ (M_2 \vee M_3) \leq_c (M_1 \circ M_2) \vee (M_1 \circ M_3)$

where $X_2 \cap X_3 = \emptyset$

(ii) $M_1 \circ (M_2 \vee^* M_3) \leq (M_1 \circ M_2) \vee^* (M_1 \circ M_3)$

where $X_2 \cap X_3 = \emptyset$

(iii) $M_1 \circ (M_2 + M_3) \leq_c (M_1 \circ M_2) + (M_1 \circ M_3)$

References

[1] S.J. Cho, J.G. Kim and W.S. Lee, "Decompositions of T-generalized transformation semigroups" (To appear in Fuzzy Sets and Systems).
[2] S.J. Cho, J.G. Kim and S.T .Kim, "On T-generalized subsystems of T-generalized state machines," Far East J. Math. Sci. 5(1), pp. 131-151, 1997.
[3] W.M.L. Holcombe, "Algebraic automata theory," Cambridge University Press, 1982.
[4] Y.H. Kim, J.G. Kim and S.J. Cho, "Products of T-generalized state machines and T-generalized transformation semigroups," Fuzzy Sets and Systems 93, pp. 87-97, 1998.
[5] D.S. Malik, J.N. Mordeson and M.K. Sen, "On subsystems of a fuzzy finite state machines," Fuzzy Sets and Systems 68, pp. 83-92, 1994.
[6] D.S. Malik, J.N. Mordeson and M.K. Sen, "Products of fuzzy finite state machines," Fuzzy Sets and Systems 92, pp. 95-102, 1997.
[7] D.S. Malik, J.N. Mordeson and M.K. Sen, "Semi-groups of fuzzy finite state machines," in: P.P. Wang, ed., Advances in Fuzzy Theory and Technology, Vol. II, pp. 87-98, 1994.
[8] D.S. Malik, J.N. Mordeson and M.K. Sen, "Submachines of fuzzy finite state machines," J. Fuzzy Math. 4, pp. 781-792, 1994.
[9] W.G. Wee, "On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification," Ph.D. Thesis, Purdue Univ., 1967.
[10] L.A. Zadeh, "Fuzzy sets," Inform. Control 8, pp. 338-353, 1965.