Simplicity and Spatiality of Involutive Quantales

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Abstract : In this note, we study recent results concerning simplicity and spatiality of involutive quantales from the algebraic point of view. When S is a sup-lattice (with a duality), $Q(S)\times Q(S^{op})$ ($Q(S)$) is an involutive quantale. Using this, some characterizations of simplicity and spatiality of involutive quantales are investigated.

1. Introduction

Involutive quantales were introduced in [11] as sup -lattices equipped with a multiplication and an involution. Such structures are well-known from the calculus of relations: the set $\mathrm{Rel}(X\times X)$ of all binary relations on any set X forms an involutive quantale. However, the motivating example of an involutive quantale is the spectrum $Max(A)$ of a non-commutative C^*-algebra A, where $Max(A)$ is the involutive quantale consisting of all closed linear subspaces of A.

The concept of a quantale is, of course, much older. It goes back to 1930's when M. Ward and R. P. Dilworth realized that the theory of ideals in rings can be conveniently formulated in terms of a lattice of ideals equipped with a multiplication ([12],[19],[20]).

An important moment in the development of the theory of quantales was the realization that quantales give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic ([5]).

In this note, we study recent results concerning simplicity and spatiality of involutive quantales from the algebraic point of view.

For facts concerning quantales in general, we refer to [16].

2. Preliminaries

A quantale is a sup-lattice equipped with an associative multiplication which distributes over arbitrary joins:

$$a\cdot \bigvee S = \quad \{a\cdot s \mid s\in S\}$$

and

$$(\bigvee S)\cdot a = \bigvee \{ s\cdot a \mid s \in S \}$$

for all $a \in L, S \subseteq L$. A quantale homomorphism is defined as a sup-lattice homomorphism between quantales preserving the multiplication.

Let x be an element of a quantale Q. Then x is called idempotent if $x\cdot x = x$ and right-sided (left-sided) if $x\cdot 1 \leq x$ ($1\cdot x \leq x$). It is called two-sided if it is both right-sided and left-sided. The set of all right-sided(left-sided, two-sided, idempotent) elements of a quantale Q is denoted by $R(Q)$ ($L(Q)$, $T(Q)$, $E(Q)$).

We shall say tha taquantale Q is right-sided(left-sided, two-sided, idempotent, right-idempotent, left-idempotent) if

$Q = R(Q)$ ($Q = L(Q), Q = T(Q), Q = E(Q), R(Q) \subseteq E(Q), L(Q) \subseteq E(Q)$).

A subset $T \subseteq Q$ is called a subquantale of Q if it is closed under all joins and multiplication in Q. It may be remarked that $R(Q)$, $L(Q)$ and $T(Q)$ are subquantales of Q closed under arbitrary meets, $Q\cdot R(Q) \subseteq R(Q)$ and $L(Q)\cdot Q \subseteq L(Q)$.

A quantale congruence on Q is an equivalence relation on Q which is a subquantale of $Q\times Q$.

A subquantale Q is said to be strong if it contains the top element of Q. The quantale Q is called trivial if $1=0$.

Since $a\cdot - : Q \to Q$ and $-\cdot a : Q \to Q$ are sup-lattice homomorphisms for any $a \in Q$, they have right adjoints $a \to_r - : Q \to Q$ and $a \to_l - : Q \to Q$, respectively. A quantale Q is discrete if

$$(a \to_l 0) \to_r 0 = a \text{ for any } a \in R(Q)$$

and

$$(a \to_r 0) \to_l 0 = a \text{ for any } a \in L(Q).$$

An involution on a quantale Q is a unary operation $*$ such that

$$a^{**} = a,$$
$$(a\cdot b)^{**} = b^{*}\cdot a^{**}$$

and

$$(\bigvee_{i\in I} a_i)^{*} = \bigvee_{i\in I} a_i^{*}$$

for all $a, b, a_i \in Q$, $i \in I$. A quantale with involution is said to be involutive. An involutive homomrphism is a quantale homomorphism of involutive quantales which also preserves the involution. An element $a \in Q$ is said to be hermitian if $a = a^{*}$.

We put $H(Q) = \{ a \in Q \mid a = a^{*} \}$. An involutive quantale Q is said to be an I-factor if $T(Q) \cap H(Q) = \{ 0,1 \}$.

3. Simplicity and Spatiality

For any sup-lattice S, we will denote by $Q(S)$ the sup-lattice of all sup-lattice endomorphisms with the pointwise ordering. Then $Q(S)$ is a quantale with multiplication given by composition. A duality on S is a unary operation $'$ such that

$$s'' = s$$
$$(\bigvee_{i\in I} s_i)' = \bigwedge_{i\in I} s_i'$$

for all $s, s_i \in Q,\ i \in I$.

Proposition 1 [11] Let S be a sup-lattice with a duality. Then $Q(S)$ is a right-idempotent discrete involutive quantale.

A non-trivial (involutive) quantale is said to be simple (I-simple) if any surjective homomorphism of (involutive) quantales from it is either an isomorphim or a constant homomorphism. Evidently, any simple involutive quantale is I-simple.

A quantale Q will be called faithful if, whenever

$$x\cdot a = y\cdot a \text{ and } b\cdot x = b\cdot y$$

for all $a\in R(Q)$ and $b\in L(Q)$, we have

$$x = y$$

for all $x, y\in Q$.

We say that the multiplication of a quantale is trivial if $x\cdot y = 0$ for each $x, y\in Q$; equivalently $1\cdot 1 = 0$. It is evident that the only simple quantale with a trivial multiplcation is the quantale 0_2. Analogously, the only I-simple quantale with a trivial multiplication is the quantale 0_2 with the identity involution.

Theorem 2 [12],[14],[15] (1) An involutive quantale with a non-trivial multiplication is I-simple if and only if it is a faithful discrete I-factor.

(2) Any I-simple involutive quantale with a non-trivial multiplication is right-idempotent.

Proposition 3 [12],[15] Let S be a sup-lattice with a duality. Then the involutive quantale $Q(S)$ is simple.

Proposition 4 [10], [15] Let Q be a simple involutive quantale which is not isomorphic to 0_2. Then Q is isomorphic to a strong involutive subquantale of $Q(S)$ for some sup-lattice S with a duality.

Theorem 5 [8] For any sup-lattice S, the quantale $Q(S) \times Q(S^{op})$ is I-simple. Moreover, any I-simple quantale is isomorphic to a strong involutive subquantale of $Q(S) \times Q(S^{op})$ for some sup-lattice S.

Proposition 6 [13] Any involutive quantale has an involutive embedding into an I-simple involutive quantale.

An (involutive) quantale Q is called (I)-spatial if, for any two distinct elements x and y of Q, there exists a strong (involutive) quantale homomorphism f into an (I)-simple quantale P such that $f(x) \neq f(y)$.

Theorem 7 [8] An involutive quantale Q satisfying $1 \cdot 1 = 1$ is I-spatial if and only if it has enough strong quantale homomorphisms from Q into quantales $Q(S) \times Q(S^{op})$.

Remark Any I-spatial involutive quantale is isomorphic to an involutive strong subquantale of a product of quantales of the form $Q(S) \times Q(S^{op})$ and of copies of 0_2.

Theorem 8 [8] Any I-spatial involutive quantale Q is spatial and, conversely, any spatial involutive quantale satisfying $1 \cdot 1 = 1$ is I-spatial.

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