Coproducts of Ringed Spaces

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Abstract : In this paper, we show that arbitrary family of ringed spaces has the coproduct and every finite coproduct of compact ringed spaces is compact.

1. Introduction

A ringed space is a topological space together with a sheaf of rings and a morphism of ringed spaces is defined by a pair of continuous map and a morphism of sheaves. It is well known that the category of rings has arbitrary products and coproducts, as the category of topological spaces has. The products in both categories are defined by the Cartesian products and canonical projections. The coproducts in the category of topological spaces are the topological sums whose underlying sets are the disjoint unions [3].

In this paper, we construct coproducts in the category of ringed spaces. We know that given a family of sheaves $\{F_i| i\in I\}$ on a topological space X the direct product sheaf on X can be constructed in a natural way [16]. We construct, for a family of topological spaces $\{X_i| i\in I\}$ and a family of sheaves $\{F_i| i\in I\}$ where each F_i is a sheaf of rings on X_i , the product sheaf of rings $\prod_{i\in I} F_i$ on the product space $\prod_{i\in I} X_i$. This idea is helpful to our work to construct the coproduct of ringed spaces. Given a family of ringed spaces, the coproduct consists of a coproduct topological space and a sheaf whose components are products of rings. Each coproduct injection consists of a coproduct injection of topological spaces and a sheaf morphism whose components are projections of rings.

Many authors have been concerned with sectional representations of rings. For example, the ring of global sections of a compact local ringed space is a Gelfand ring [3]; the ring of global sections of a compact simple ringed space is a biregular ring [2]; the ring of global sections of a compact prime ringed space is a weakly Baer ring [15]. A common feature of the ringed spaces in above examples is the compactness. Our interest is the properties of the category of compact ringed spaces. The compactness of a topological space X may be described in terms of the ring $\mathbb{R}(X)$ of continuous real valued functions on X. A ringed space (X, F) is compact when the topological space X is compact and the sheaf F has some properties similar to those of $\mathbb{R}(X)$. We state and prove the analogue of the property that topological products of compact spaces are compact.

2. Preliminaries

A *presheaf* F of *sets* on a topological space X is a functor $\Omega(X)^{op} \rightarrow$ ***Set*** where $\Omega(X)$ is the partially odered set of open sets in X. For each pair of open sets $U \supset V$ of X, the morphism $F(U) \rightarrow F(V)$ is denoted by ρ_V^U . A *morphism* of presheaves $F(U) \rightarrow F(V)$ is defined by a natural transformation.

For a presheaf F on X and $x \in X$, the *stalk* F_x of F at x is defined by the direct limit $\varinjlim F(U)$, where U runs through all open neighborhoods of x in X. For each $s \in F(U)$ with U an open neighborhood of x, the image of s under the coproduct injection is denoted by s_x

A presheaf F of sets on X is called a *sheaf* if it satisfies the following condition:

For any open covering $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ with open sets U, U_λ in X, the sequence of maps

$$F(U) \rightarrow \prod_{\lambda \in \Lambda} F(U_\lambda) \rightrightarrows \prod_{(\mu, \nu) \in \Lambda \times \Lambda} F(U_\mu \cap U_\nu)$$

is an equalizer diagram.

A *sheaf space* over X is a pair (E, p) of a topological space E and a surjective local homeomorphism $p: E \rightarrow X$.

Let $LF = \coprod_{x \in X} F_x$, the disjoint union of stalks, together with the topology generated by the sets $\hat{s}(U) = \{s_x \mid s \in U\}$ for all open sets U in X and $s \in F(U)$, where $\hat{s}$ is the map $U \rightarrow LF$, $x \mapsto s_x$ Then each $\hat{s}$ is continuous and the natural projection $p: LF \rightarrow X$, $s_x \mapsto x$ is a surjective local homeomorphism. Hence (LF, p) is a sheaf space over X.

Conversely, for each sheaf space (E, p), a sheaf (of sets) ΓE can be constructed as follows:

For each open set U in X,

$$\Gamma E(U) = \{\sigma : U \rightarrow \mid \sigma \text{ is continuous and } p\sigma = id_U\}$$

and for each pair of open sets $U \supset V$ in X,

$$\rho_V^U : \Gamma E(U) \rightarrow \Gamma E(V), \quad \sigma \mapsto \sigma|_V$$

Each $\sigma \in \Gamma E(U)$ is called a *section* (of p) and ρ_V^U is called a *restriction map.*

For a presheaf F, the sheaf ΓLF is called the *sheafification* of F.

Lemma 2.1 Let (E, p) be a sheaf space over X. Then, for each $x \in X$, $p^{-1}(x) = (\Gamma E)_x$

Proposition 2.2 If E is a sheaf space over X, then $L\Gamma E \cong E$

Lemma 2.3 If F is a sheaf on X, then $F \cong \Gamma LF$.

Now if F is a sheaf then we can identify F with the sheafification ΓLF. $F(U)$ is called the set of *sections*. In particular, sections in $F(X)$ are called *global sections*. For each section $\sigma \in F(U)$ and open subsets $V \subset U$, the map ρ^U_V is called the *restriction* and we write $\sigma|_V$ instead of $\rho^U_V(\sigma)$.

A *sheaf of rings* on a topological space X is a sheaf space (E, p) over X satisfying the following conditions:

(1) for each $x \in X$, $p^{-1}(x)$ is a ring;

(2) the ring operations

$$E \times_p E = \{(a, b) \mid p(a) = p(b)\} \to E, \quad (a, b) \mapsto a-b$$

and

$$E \times_p E \to E, \quad (a, b) \mapsto ab$$

are continuous;

(3) the assignment of identity $X \to E$, $x \mapsto 1_x \in p^{-1}(x)$ is continuous.

Remark By Proposition 2.2 and Lemma 2.3, we may redefine a sheaf of rings on X by a sheaf F of sets on X such that, for each open set U in X, $F(U)$ is a ring and the restriction maps ρ^U_V are ring homomorphisms.

If $f: X \to Y$ is a continuous map and F is a sheaf on X then the *direct image sheaf* of f, denoted by f_*F, is a sheaf on Y defined by

$$f_*F(V) = F(f^{-1}(V)) \text{ and } \rho^V_{V'} = \rho^{f^{-1}(V)}_{f^{-1}(V')}$$

for each pair of open sets $V' \subset V$ in Y.

3. Ringed spaces

Definition 3.1 A *ringed space* is a pair (X, F) of topological space X and a sheaf F of rings on X. Given ringed spaces (X, F) and (Y, G) a *morphism* $(X, F) \to (Y, G)$ is a pair $(f, f^\#)$ of a continuous map $f: X \to Y$ and a sheaf morphism $f^\#: G \to f_*F$.

Given morphisms $(f, f^\#): (X, F) \to (Y, G)$ and $(g, g^\#): (Y, G) \to (Z, H)$ of ringed spaces, the composition is defined by the pair $(g \circ f, (g \circ f)^\#)$ of the composition of maps $g \circ f$ and the sheaf morphism $(g \circ f)^{\#} : H \to (g \circ f)_{*}F$ which assigns, to each open set W in Z, the ring homomorphism

$$(g \circ f)^{\#}(W) : H(W) \to (g \circ f)_{*}F(W) = f_{*}F(g^{-1}(W)) = F(f^{-1} \circ g^{-1}(W))$$

such that

$$f^{\#}(g^{-1}(W)) \circ g^{\#}(W) = (g \circ f)^{\#}(W).$$

Theorem 3.2 The category of ringed spaces has arbitrary coproducts.

Definition 3.3 A ringed space (X, F) is said to be *Hausdorff* if and only if, for any distinct elements x_1, x_2 of X, there is a global section $\sigma \in F(X)$ such that $\sigma(x_1) = 1$ and $\sigma(x_2) = 0$

Definition 3.4 A ringed space (X, F) is said to be *Completely regular* if and only if, for any $x \in X$ and closed subset B of X not containing x, there is a global section $\sigma \in F(X)$ such that $\sigma(x_1) = 1$ and $\sigma|_B = 0$

Definition 3.5 A ringed space (X, F) is said to be *Compact* if and only if it is completely regular and X is a compact topological space.

Clearly, every completely regular ringed space is Hausdorff. The converse is proved in [14]. Thus, we have :

Proposition 3.6 A ringed space (X, F) is Compact if and only if it is Hausdorff and X is compact.

Theorem 3.7 Finite coproducts of compact ringed spaces are compact.

References

[1] J. Dauns and K. H. Hofmann, The Representation of Biregular Rings by Sheaves, Math. Z, 91(1966), 103-123.

[2] J. Dugundji, Topology, Allyn and Bacon, 1966.

[3] M. Eisenberg, Topology, Holt, Rinegart and Winston, Inc, 1974.

[4] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977.

[5] H. Herrich and G. E. Strecker, Category Theory, Allyn and Bacon, Boston, 1973.

[6] K. H. Hofmann, Representations of Algebras by Continuous Sections, Bull. Amer. Math. Soc. (1972), 291-373.

[7] F. Kasch, Modules and Rings, Academic Press, 1982.
[8] I. G. MacDonald, Algebraic Geometry: Introduction to Schemes, Benjamin, New York, 1968.
[9] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, New York, 1971.
[10] I. Moerdijk and S. Mac Lane, Sheaves in Geometry and Logic: A First Introduction to Topos Theory, Springer-Verlag, New York, 1992.
[11] C. J. Mulvey, Compact Ringed Spaces, J. Algebra 52(1978), 411-436.
[12] J. R. Munkres, Topology: A First Course, Prentice-Hall, 1975.
[13] L. Rowen, Ring Theory, Academic Press, 1988.
[14] S. Sun, On Biregular Rings and Their Duality, J. Pure Appl. Algebra, 89(1993), 329-337.
[15] S. Sun, Duality on Compact Prime Ringed Spaces, J. Algebra 169(1994), 805-816.
[16] B. R. Tennison, Sheaf Theory, Cambridge University Press, 1975.