# Modeling and Scheduling of Cyclic Shops with Time Window Constraints

Jeong-Won Seo and Tae-Eog Lee
Department of Industrial Engineering
Korea Advanced Institute of Science and Technology
E-mail: telee@mail.kaist.ac.kr

ABSTRACT

A cyclic shop is a production system that repeatedly produces identical sets of jobs, called minimal part sets, in the same loading and processing sequence. We consider a version of cyclic shop where the operations are processed and unloaded within time limits, so called a time window. We model the shop using an event graph model, a class of Petri nets. To represent the time window constraint, we introduce places with negative time delays. From the shop modeling graph, we develop a linear system model based on the max- plus algebra and characterize the conditions on the existence of a stable schedule.

Key words: cyclic scheduling, time window constraints, event graph, max-plus algebra.

## 1. Introductions

Manufacturers have tried to reduce work-in-progress (WIP) and manufacturing lead times, especially when there are demands on multiple product models. Therefore, setup times for switching product models at a machine have been significantly reduced by automation or process innovation. Examples include automated flexible manufacturing systems, cellular lines and flexible lines. Such a flexible shop can simultaneously produce multiple product models. In this case, a cyclic production method of producing multiple items in a cyclic order is often used. For cyclic production, the production requirement of multiple items is decomposed into the smallest set that has the same or approximately same proportion of items, called a *minimal part set* (MPS) [10]. For example, when the production requirement is given as 5000 units of part A, 3000 units of part B, and 2000 units of part C, the MPS is {5A, 3B, 2C}. The MPSs are produced 1000 times in the same loading and processing sequence at each machine. Such a flexible shop is called a *cyclic shop* and the scheduling method is called *cyclic scheduling.* Recent industrial examples of cyclic production include flexible manufacturing systems, PCB assembly lines, video cassette recorder deck assembly lines, refrigerator assembly lines and even heavy equipment assembly lines. Especially, a shop with chemical or plating treatments, such as an automated electroplating line for producing printed circuit boards and a cluster tool for wafer fabrication, has the critical timing constraints which the jobs are processed and unloaded within a time limit, so called *time window constrains*. Generally, the time windows are defined as a pair of upper and lower bound of the sojourn time in the machines (or processors), that is, they are given as $[d_i, d_i+\delta_i]$, where $d_i$ denotes a processing time of operation $i$ and $\delta_i$ denotes a (maximal) waiting time after the process is completed.

A cyclic scheduling problem is to determine the processing sequence and the starting times of the operations at each machine when the MPS is given and the operations are assigned to the machines. In a shop with time window constraints, there is two important issues. One is to determine whether the sequence is feasible, that is, in a given sequence, there exists starting times of operations that satisfy the time window constraints. The other is to find the starting times of each operation so that the operations are finished within the time windows. In the cyclic scheduling problem, the primary performance measure is the *cycle time* (or its reciprocal, *throughput rate*).

There are studies on versions of cyclic scheduling problems including Ahmadi and Wurgaft [1], Chaar and Davidson [3], Graves et al. [6], Hanen [7], Hall et al. [8, 9], Kamoun and Sriskandarajah [11], Karabati and Kouvelis [12], Lee [14], Lee and Posner [13], Matsuo [15], McCormick et al. [16], Roundy [17], and Sethi et al. [18]. Most of them discuss sequencing and timing issue for cyclic shops without time window constraints.

For a basic cyclic job shop, Lee [14] propose the use of a linear system approach based on a special algebra of [4] called the max-plus algebra for proving the existence of SESSs and developing an efficient algorithm of computing all SESSs.

We are interested in extending the results of Lee [14] to cyclic shops with time window constraints. Once we can develop a linear system model for this shop, we would prove the existence of *stable timely starting schedules* (STSSs) that each operation starts when all its preceding operations are completed and the time window constraint is satisfied.

In this paper, we discuss the scheduling problem and the timing control of an cyclic shop with time window constraints. We characterize the condition on the existence of STSSs. To do this, we introduce a modified event graph model, which has negative time delay. We then develop a linear system model based on the max-plus algebra using the modified event graph model.

## 2. An Event Graph Model

We define an cyclic shop with time window constraint. Let $M$ be the set of machines (or workstations) and $J$ be the set of jobs that comprises an MPS. There can be multiple jobs of the same type in the MPS. However, without loss of generality, we assume that there are $J$ "distinct" jobs in the MPS. Each job has one or more operations that have some technological precedence constraints among them. Let $N$ be the set of operations in the MPS. A machine repeats the assigned operations in an identical cyclic order. Each job has a time limit in each machine, that is, operation $i$ completed a process has to be unloaded from the machine within $\delta_i$, called a maximal waiting time. We model the shop using an event graph with negative time delays, a class of Petri nets. For simplicity, we assume that the jobs are immediately transported to the next machine. We also model the tasks of material handling systems, such as robots, AGVs, or hoists.

To model a cyclic shop with time window constrains, we first model the starting (loading) and the ending (unloading) of each operation. Second, the precedence relationship and the time window constrains are modeled arcs and time delays. For example, we let $x_i$ and $x_j$ be a starting time and an ending time of an operation. Then, the precedence relationship is given as $x_j \geq x_i + p_i$, where $d_i$ is the processing time of operation $i$. And, the time window constraint is given as $x_j \geq x_i + p_i + \delta_i$, where $\delta_i$ is the maximal time limit by which the job waits in the machine after processing. This two relationships for each operation are modeled as circuits with negative circuit weights. In Example 1, we illustrate a modified event graph model for a cyclic flow line with time window constrains.

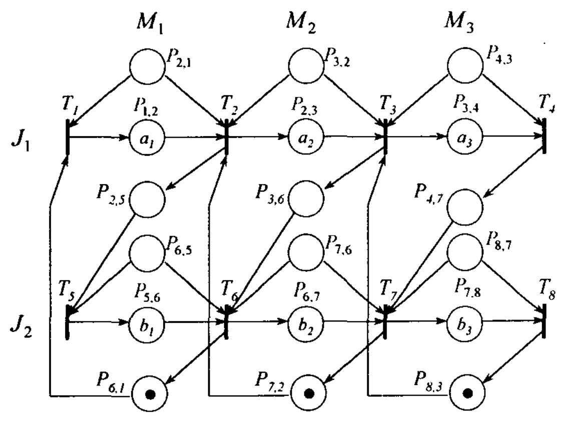


Figure 1. An event graph model for a cyclic flow line with time window constraints.

**Example 1.** The shop has three machines, $M_1$, $M_2$, and $M_3$, that process two parts, and $J_2$. The MPS is ( $1J_1, 1J_2$). Part $J_1$ and $J_2$ have operations, ( $a_1, a_2, a_3$) and ( $b_1, b_2, b_3$), respectively. The input sequence is $J_1$ and $J_2$. The precedence relations of the operations and the time window constraints are shown as a graph in Figure 1.

In Figure 1, the bold bars are immediate transitions with no time delay. The immediate transitions represent the tasks of loading and unloading tasks. For example, $T_1$ represents a loading task of operation $a_1$ into machine $M_1$. $T_2$ represents both the unloading task of operation $a_1$ from machine $M_1$ and the loading task of operation $a_2$ into machine $M_2$. Place $P_{ij}$ on a machine cycle represents an operation $i$ and has an operation time $d_i$. Place $P_{ji}$ represents the time window constraint and has a negative time delay, $-d_i - \delta_i$. For example, $P_{12}$ is a timed place modeling operation $a_i$ and $P_{21}$ is a timed place with a negative time delay $-d_{a_1} - \delta_{a_1}$. In each machine circuit, the place having a token, called *recycling place*, initiates the next MPS instance. Such a modified event grap model is denoted by *PN*.

## 3. A Steady State Analysis

For timing control of the operations, we take the *timely starting* strategy that each operation starts when all its preceding operations are completed and the time window constraints are satisfied. Because of the time window constraints, some operations start after appropriate time delays in order to be completed within time limits of the processes. Therefore, the timely starting strategy need to deliberately control the timings of the operations and to determine the appropriate time delays of the operations.

For the timely starting strategy, we analyze the schedule pattern or the timing pattern of the operations and to know whether the schedule repeats an identical pattern for each MPS instance.

The steady state, if it exists, is defined as $x_i^{r+1} - x_i^r = \lambda$ for all $r = 1, 2, \ldots$, where $x_i^r$ is the timely epoch time of transition $i$ at the $r$-th MPS instance and $\lambda$ is the steady state *cycle time.* A schedule in a steady state is called *stable* and $\lambda$ is called the *steady state cycle time* or simply *cycle time.* We wish to know whether there exists a *stable timely staring schedule* (STSS) such that the cycle time $\lambda$ is minimal for the given processing sequence of the operations at each machine and how we can identify such STSSs. Once existence of STSSs is identified, the scheduling problem reduces to the scheduling problem for a single MPS.

To analyze the schedule pattern, we first develop a dynamic equation for the timely starting times based on the shop modeling graph. $N_\alpha \subseteq N$ and $N_\beta \subseteq N$ denote the sets of the first transitions and the last transitions of the MPS on each machine, respectively. $N_m$ denotes the transitions associated with loading and unloading tasks to machine $m$. Then, we have the following dynamic equation.

***DE***: For $r=1,2,\ldots,$

$$x_j^{r+1}=\begin{cases}\max\{x_k^{r+1}+d_{kj},x_i^r+d_{ij}\}, \\ \quad \text{if } j\in N_\alpha, i\in N_\beta, i,j\in N_m \\ \max\{x_k^{r+1}+d_{kj},x_i^{r+1}+d_{ij}\}, \\ \quad \text{otherwis,}\end{cases}$$

where $d_{ij}$ is the delay time of place $P_{ij}$, and $k$ is the transition preceding $j$ in the same job flow. Since $k$ precedes $j$ in the same job flow, $k$ and $j$ has the same MPS instance. If $j$ is the first transition of a job, then the preceding operation $k$ does not exist. In this case, $k$ is considered as a dummy transition 0 such that $d_{0j}=-\infty$. The first part of ***DE*** states that when $j^{r+1}$ is the first transition of ($r$+1)-th MPS at a machine, it can start after the last operation of the previous MPS instance as well as its preceding operation $k^{r+1}$ in the same MPS instance.

To analyze steady state behavior of ***DE***, we develop a linear system model for ***DE***. Then, we show that the linear system model has a finite eigenvalue and eigenvectors, which define the steady state cycle time and the steady states of ***DE***, respectively. To represent ***DE*** in a linear form, we use the max-plus algebra of Baccelli et al. [2] and Cuninghame-Green [4] (for more details, see the references).

Now, we define the matrices for modeling ***DE*** for the modified event graph. We let A and B denote $n\times n$ matrices with elements $a_{ij}$ and $b_{ij}$, respectively. $a_{ij}=d_{ij}$ and $b_{ij}=0$. Using $(R,\oplus,\otimes)$, ***DE*** is then represented in a matrix form:

$$x^{r+1}=(x^{r+1}\otimes A)\oplus(x^r\otimes B)$$

where $x^r\equiv(x_1^r,x_2^r,\ldots,x_n^r)$.

Matrix A is considered as the incidence matrix of the subgraph of the modified event graph that excludes the recycling places $P_{ij}$ and the associated arcs at each machine, where $j\in N_\alpha, i\in N_\beta$. $A_{ij}^k$ is the length of the longest path among all paths from node $i$ to node $j$ with exactly $k$ arcs in the subgraph. In our modified event graph, the subgraph has some circuits because of the places and arcs for time window constraints. Therefore, $A^*(\equiv I\oplus A\oplus A^2\cdots\oplus A^\infty)$ may diverse when a circuit of the subgraph has a positive circuit weight (see Theorem 3.20 of [2]).

**Lemma 1**. *If the subgraph has no circuits with positive circuit weights, then matrix* $\hat{A}=B\otimes A^*$ *is finitely defined. And, we have a linear system*:

$$x^{r+1}=x^r\otimes\hat{A},\quad r=1,2,\ldots$$

*The linear system relates the timely starting times of the operations, by which we can analyze the schedule pattern of each MPS.*

**Proof.** The proof is directly derived by Theorem 3.20 of [2].

Using Lemma 1, we can verify whether the shop has a feasible cyclic schedule. And, we can determine the time limits for each operation in order to produce the job within the time window.

**Example 2**. In the Example 1, we suppose that $d_{a_1}=4$, $d_{a_2}=3$, $d_{a_3}=2$, $d_{b_1}=2$, $d_{b_2}=5$, $d_{b_3}=1$, and $\delta_i=0$ for all operations. This implies that the shop has no-wait constraints. Then, we have matrices A , B and $\hat{A}$ as follows.

$$A=\begin{pmatrix}-\infty&4&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-4&-\infty&3&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-3&-\infty&2&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-2&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&2&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-2&-\infty&5&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-5&-\infty&1\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-1&-\infty\end{pmatrix},$$

$$B=\begin{pmatrix}-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\0&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&0&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&0&-\infty&-\infty&-\infty&-\infty&-\infty\end{pmatrix}$$

$$\hat{A}=\begin{pmatrix}-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty&-\infty\\0&4&7&9&5&7&12&13\\-4&0&3&5&1&3&8&9\\-7&-3&0&2&-2&0&5&6\end{pmatrix}$$

We have eigenvalue and eigenvector as follows, $\lambda=8$ and $x=(-4,0,3,5,3,8,9)$. The eigenvector define a stable timely starting schedule. The STSS is ad follows.

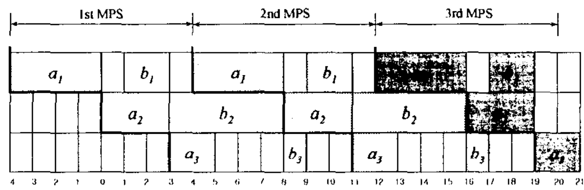


Figure 2. A STSS of Example 1

## 4. Final Remarks

We have characterized the steady states of an cyclic shop with time window constraints. To do this, we have adapt the results of linear system theory. By introducing a modification of the Petri net model, we have modeled the time window constraints and were able to identify the cycle time and the STSSs of the shop.

It remains to develop an optimization model for sequencing using the steady state results. We would apply and extend this model to the shop with material handling system such as robot, AGVs, or hoist.

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