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Free Vibrations of Stepped Horizontally Curved Beams

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Abstract

The differential equations governing the free vibrations of stepped horizontally circular curved beams with circular cross-section are derived and solved numerically. In numerical method, the Runge-Kutta and Determinant Search methods are used for computing the natural frequencies and mode shapes. Frequencies and mode shapes are reported as the functions of non-dimensional system parameters. The numerical method developed herein for computing frequencies and mode shapes are efficient and reliable.

Keywords: curved beam, free vibration, natural frequency, mode shape, stepped beam

1. INTRODUCTION

Studies on the free vibrations of linearly elastic curved beams of various geometries have been reported by many investigators.[1] [4] However, the works on the free vibration of horizontally curved beam with variable cross-section, especially stepped curved beams, are very rare. The main purpose of the present paper is, therefore, to present numerical method for calculating the natural frequencies and mode shapes of stepped horizontally curved beams.

The differential equations were derived for free, out-of-plane vibration of linearly elastic circular curved beam with variable cross-section, and the stepped curved beam with circular cross-section was applied into this differential equations. The governing differential equations were solved numerically to obtain the natural frequencies and mode shapes. In numerical method, the Runge-Kutta and the Determinant Search methods were used to integrate the differential equations and calculate the natural frequencies, respectively. In numerical examples, the curved beams with both clamped ends and both hinged ends were considered. The four lowest natural frequencies were presented as functions of the non-dimensional system parameters.

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2. MATHEMATICAL MODEL

The geometry of stepped horizontally circular curved beam is defined in Figure 1. Both ends are either clamped or hinged. Its radius and opening angle are a and α, respectively. The radial line to a typical beam point is inclined at angle θ with the radial line of left end. Also shown in Figure 1 are positive direction of vertical displacement v, positive directions of rotation ψ and β of cross-section due to the bending moment and shear force, respectively, and positive direction of angle of twist ϕ. The beam is sectored into several segments for the stepped cross-section. The segment ratio which is defined as a ratio of the subtended angle of ith segment to the opening angle α is depicted as m_i. The thickness, cross-sectional area, area moment of inertia and torsional constant of cross-section of the ith segment are D_i, A_i, I_i and J_i($i=1,2,3,\cdots,n$), respectively.

The quantities A_i, I_i and J_i of the ith segment are expressed in the form

$$A_i=A_1 f_i,\quad I_i=I_1 g_i,\quad J_i=J_1 h_i \tag{1-3}$$

where $f_i=f(\theta)$, $g_i=g(\theta)$ and $h_i=h(\theta)$ are the functions of the single variable θ in the ith segment as discussed later in shape functions of the section 3.

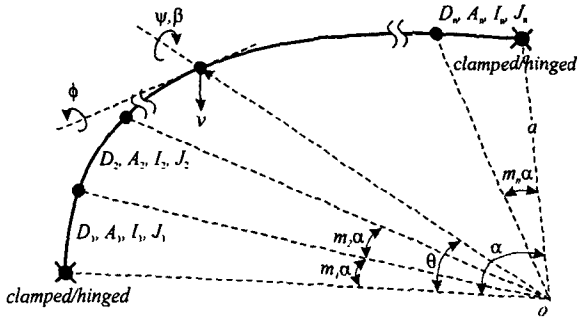


Fig. 1 Geometry of curved beam

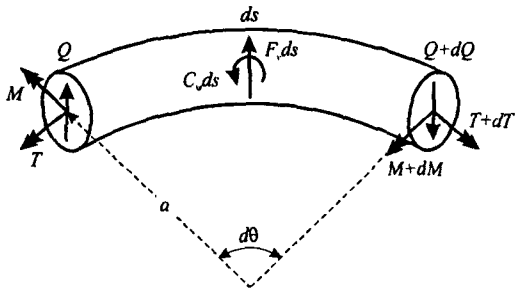


Fig. 2 Loads on a curved beam element

A small element of the curved beam with the opening angle $d\theta$ and arc length ds shown in Figure 2 defines the positive directions for the shear force Q, the bending moment M, the torsional moment T, the vertical inertia force F_v and the rotatory inertia couple C_ψ. With the inertia force and the rotatory inertia couple treated as equivalent static quantities, the three equations for dynamic equilibrium of element are

$$\frac{dQ}{d\theta}-aF_v=0,\quad \frac{dM}{d\theta}-aQ+T+aC_\psi=0,\quad M-\frac{dT}{d\theta}=0 \tag{4-6}$$

The equations that relate M and T to the rotations ψ and ϕ are as follows.[5]

$$M=\frac{EI}{a}\left(\phi-\frac{d\psi}{d\theta}\right)=\frac{EI_1g}{a}\left(\phi-\frac{d\psi}{d\theta}\right),\ T=\frac{GJ}{a}\left(\psi+\frac{d\phi}{d\theta}\right)=\frac{GJ_1h}{a}\left(\psi+\frac{d\phi}{d\theta}\right) \tag{7-8}$$

where E and G are the Young's modulus and shear modulus, respectively.

The transverse shear force Q, whose effect on the structural behavior is known as the effect of shear deformation, is given by[6]

$$Q=k\beta GA=kGA_1f\left(\frac{1}{a}\frac{dv}{d\theta}-\psi\right) \tag{9}$$

where k is the cross-sectional shape factor. For example, the k values for the rectangular and circular sections are 2/3 and 3/4, respectively.

The beam is assumed to be in harmonic motion, or each co-ordinate is proportional to $\sin(\omega t)$ where ω is the frequency parameter and t is time. The inertia loadings per unit arc length are then

$$F_v=-dA\omega^2v=-dA_1f\omega^2v,\ \ C_\phi=-dIa\omega^2\psi=-dI_1ga\omega^2\psi \tag{10,11}$$

where d is mass density of beam material. The C_ϕ term of equation (11) is known as the effect of rotatory inertia in free vibrations.

To facilitate the numerical studies, the following non-dimensional system parameters are defined.

$$\eta=v/a,\ \ \lambda=\frac{a}{\sqrt{I_1/A_1}},\ \ \gamma=kG/E,\ \ \varepsilon=GJ_1/(EI_1),\ \ p_i=\omega_ia^2\sqrt{dA_1/(EI_1)} \tag{12-16}$$

The vertical displacement v is normalized by the curved beam radius a and the variables of λ, γ, ε and p_i are the slenderness ratio, shear parameter, stiffness parameter and frequency parameter, respectively.

When equations (9) and (10) are substituted into equation (4) and the non-dimensional forms of equations (12)-(16) are used, the result is

$$\frac{d^2\eta}{d\theta^2}=-\frac{1}{f}\frac{df}{d\theta}\frac{d\eta}{d\theta}-\frac{p_i^2}{\gamma\lambda^2}\eta+\frac{d\psi}{d\theta}+\frac{1}{f}\frac{df}{d\theta}\psi \tag{17}$$

When equations (7), (8), (9) and (11) are substituted into equation (5) and the non-dimensional forms of equations (12)-(16) are used, the result is

$$\frac{d^2\psi}{d\theta^2} = -\gamma\lambda^2\frac{f}{g}\frac{d\eta}{d\theta} - \frac{1}{g}\frac{dg}{d\theta}\frac{d\psi}{d\theta} + \left(\gamma\lambda^2\frac{f}{g} + \varepsilon\frac{h}{g} - \frac{p_i^2}{\lambda^2}\right)\psi$$
$$+\left(1+\varepsilon\frac{h}{g}\right)\frac{d\phi}{d\theta} + \frac{1}{g}\frac{dg}{d\theta}\phi \tag{18}$$

When equations (7) and (8) are substituted into equation (6) and the non-dimensional forms of equations (12)~(16) are used, the result is

$$\frac{d^2\phi}{d\theta^2} = -\left(1+\frac{g}{\varepsilon h}\right)\frac{d\psi}{d\theta} - \frac{1}{h}\frac{dh}{d\theta}\left(\psi + \frac{d\phi}{d\theta}\right) + \frac{1}{\varepsilon}\frac{g}{h}\phi \tag{19}$$

For the beam with both clamped ends, the boundary conditions at ends $\theta = 0$ and $\theta = \alpha$ are

$$\eta = 0,\ d\eta/d\theta = 0,\ \phi = 0 \tag{20-22}$$

For the beam with both hinged ends, the boundary conditions at ends $\theta = 0$ and $\theta = \alpha$ are

$$\eta = 0,\ d\psi/d\theta = 0,\ \phi = 0 \tag{23-25}$$

3. SHAPE FUNCTIONS: f, g and h

The shape functions f, g and h first introduced in equations (1), (2) and (3), and contained in the governing differential equations (17), (18) and (19), are now defined. The examples are limited circular cross-section whose diameter of ith segment is D_i mentioned above already. A non-dimensional system parameter defined as the section ratio is introduced newly as follows.

$$n_i = D_i/D_1 \tag{26}$$

in which it is clear that the value of n_1 is one because of $n_1 = D_1/D_1 = 1$.

With equations (1), (2), (3) and (26), the f_i, g_i and h_i of the circular cross-section for the ith segment are expressed in term of section ratio n_i as follows.

$$f_i = n_i^2,\ g_i = h_i = n_i^4 \tag{27,28}$$

When equations (27) and (28) are differentiated once, the results are

$$df/d\theta = 0,\ dg/d\theta = dh/d\theta = 0 \tag{29}$$

4. NUMERICAL METHOD AND COMPUTED RESULTS

Based on the above analysis, a general FORTRAN computer program was written to calculate p_i, $\eta = \eta_i(\theta)$, $\psi = \psi_i(\theta)$ and $\phi = \phi_i(\theta)$. The numerical method described by Oh et al.[7] was used to solve the differential equations (17)-(19), subjected to the end constraint equations (20)-(22) or (23)-(25). The curved beams with both clamped ends and both hinged ends were considered for given parameters α, m_i, n_i, λ, γ and ε. First, the Runge-Kutta method was used to integrate the differential equations; and then the Determinant Search method was used to calculate the characteristic values p_i.

In this study, the shear parameter γ and stiffness parameter ε were chosen as 0.29 and 0.77, respectively, since the cross-section was circular with $k = 3/4$ and $J_1 = 2I_1$, and the material was assumed to be steel with $G/E = 0.385$. The four lowest values of p_i with their corresponding mode shapes were calculated.

For numerical examples, suitable convergence of solutions was obtained for an increment of $\Delta\theta = \alpha/50$ in Runge-Kutta method. The convergence criterion was that p_i solutions obtained with the $\alpha/50$ increment agreed with those obtained with the $\alpha/300$ increment to within three significant figures. The numerical results, given in Table 1 and Figures 3-7, are summarized as follows.

For the comparison purposes, the finite element solutions based on the commercial package SAP 90 were used to compute the frequency parameters p_i. The results are shown in Table 1 in which the frequencies of this study agree closely with those of SAP 90 within a tolerance of 3%.

Table 1 Comparison of frequency parameters p_i between this study and SAP 90

($\alpha = 60°$, $m_1 = m_2 = m_3 = 0.33$, $n_2 = 1.2$, $n_3 = 1$, $\lambda = 80$, $\gamma = 0.29$, $\varepsilon = 0.77$)

End constraints	i	Frequency parameter, p_i		Deviation (%)
		this study(A)	SAP 90(B)	|B-A|/A
both clamped ends	1	19.89	20.09	1.01
	2	58.32	59.97	2.83
	3	112.1	110.4	1.52
	4	179.5	180.9	0.78
both hinged ends	1	7.528	7.589	0.81
	2	36.98	36.26	1.94
	3	81.34	82.18	1.03
	4	142.2	141.2	0.70

It is shown in Figure 3, for which both clamped ends and both hinged ends with $\alpha = 60°$, $m_1 = m_2 = m_3 = 0.33$, $n_3 = 1$, $\lambda = 80$, $\gamma = 0.29$, $\varepsilon = 0.77$ that the frequency parameters p_i increase as the section ratio n_2 increases to 3. It is observed that the increasing rate of p_i vs. n_2 curves is higher at higher mode. Particularly, the increasing rate of first mode is negligible.

It is shown in Figure 4, for which both clamped ends and both hinged ends with $\alpha=60^{o}$, $m_1=m_3=(1-m_2)/2$, $n_2=1.2$, $n_3=1$, $\lambda=80$, $\gamma=0.29$, $\varepsilon=0.77$, that the p_i values increase as the segment ratio m_2 increases. It is noted that in case of the first mode of the both hinged ends, the effect of m_2 on p_i is negligible.

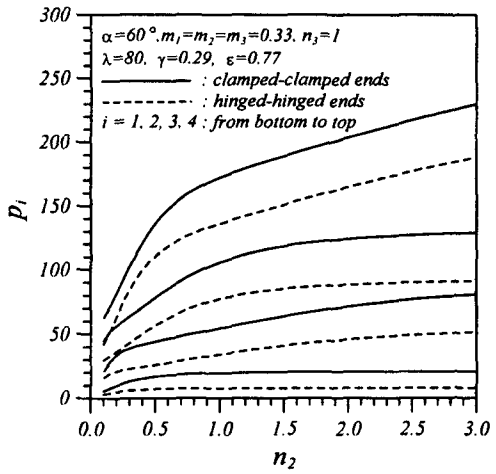


Fig. 3 p_i vs. n_2 curves

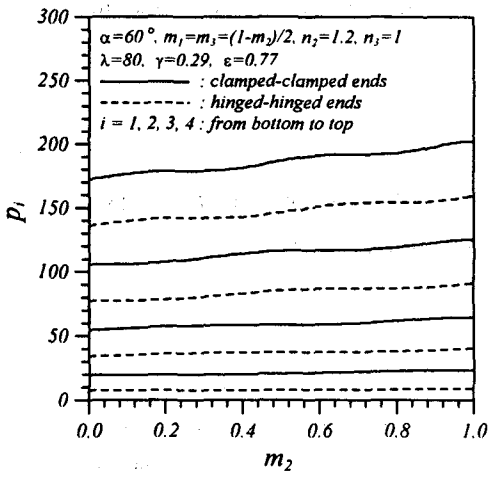


Fig. 4 p_i vs. m_2 curves

It is shown in Figure 5, for which both clamped and both hinged beams with $\alpha=60^{o}$, $m_1=m_2=m_3=0.33$, $n_2=1.2$, $n_3=1$, $\gamma=0.29$, $\varepsilon=0.77$, that the p_i values increase and approach upper limits or horizontal asymptotes as the slenderness ratio λ increases to 200. It is noted that in case of the first modes, the effect of λ on p_i is negligible.

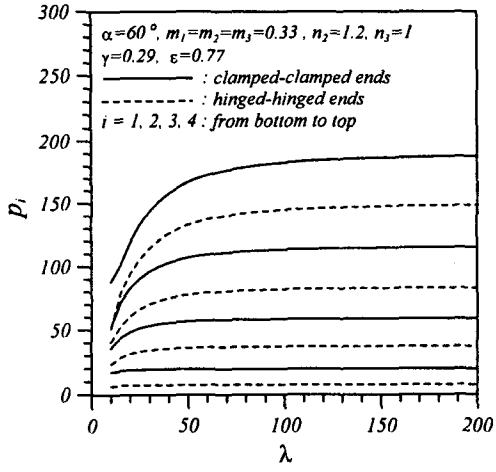


Fig. 5 p_i vs. λ curves

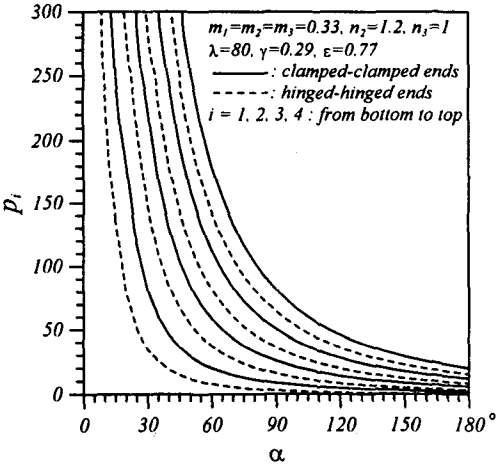
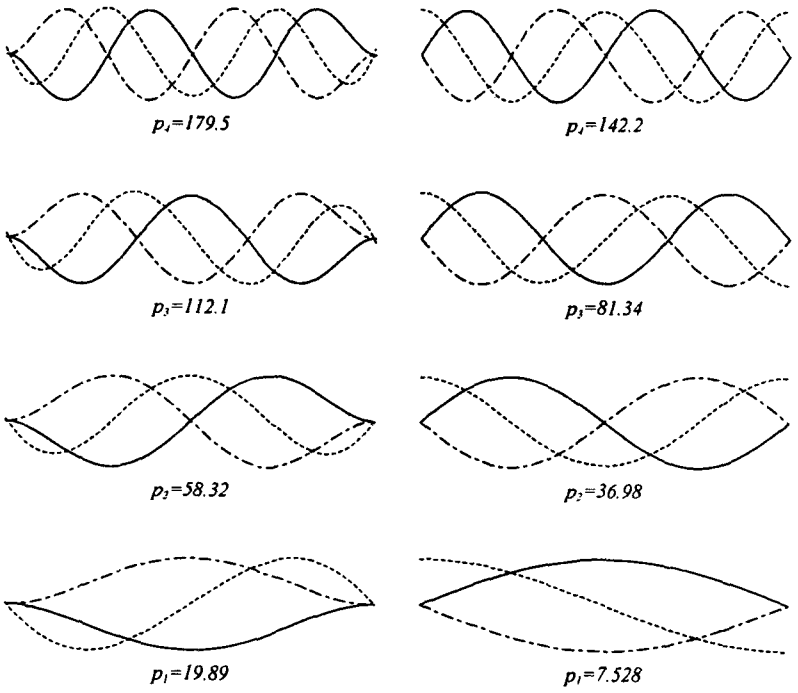


Fig. 6 p_i vs. α curves

In Figure 6, both clamped ends, both hinged ends, $m_1 = m_2 = m_3 = 0.33$, $n_2 = 1.2$, $n_3 = 1$, $\lambda = 80$, $\gamma = 0.29$, $\varepsilon = 0.77$ and the p_i values all decrease very rapidly as the opening angle α increases from 30° to 90°. And it is true that the p_i values approach lower limits or horizontal asymptotes as α increases to 180°.

Typical mode shapes are shown in Figure 7, based on both clamped ends and both hinged ends, $m_1 = m_2 = m_3 = 0.33$, $n_2 = 1.2$, $n_3 = 1$, $\lambda = 80$, $\gamma = 0.29$, $\varepsilon = 0.77$, $\alpha = 60^\circ$.



(a) both clamped ends (b) both hinged ends

Fig. 7 Example of mode shapes($\alpha = 60^\circ$, $m_1 = m_2 = m_3 = 0.33$, $n_2 = 1.2$, $n_3 = 1$, $\lambda = 80$, $\gamma = 0.29$, $\varepsilon = 0.77$; ——— : v, - - - - - - - - : ψ, - · - · - · - · - : ϕ)

6. CONCLUSIONS

The method presented here for calculating frequencies and mode shapes for stepped horizontally circular curved beams were found to be efficient and reliable over a wide range of system parameters. Governing differential equations for free vibration of such curved beam were derived and solved numerically. Computations showed that the frequencies obtained by present study and SAP 90 agreed closely. The lowest four frequency parameters are presented as functions of four non-dimensional system parameters.

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