A note on fuzzy Fell topology

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1. preliminaries

For a set X, I^X denote set of all fuzzy sets, where $I=[0,1]$ and if $A \in I^X$, then $A: X \to I$(See [9]). A fuzzy set A in X is called a *fuzzy point* in X with spport x and value $\lambda \in (0,1]$, denoted by x_λ, if for each $y \in X$

$$A(y_\lambda) = \begin{cases} \lambda & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X will be denoted by $F_p(X)$ (See[7]).

The concepts of fuzzy continuity, F-open(resp. F-closed)mapping and fuzzy product space refers to [2], [1] and [8] respactively. Futhermore, the concepts of quasi-coincident and separation axioms refers to [7] and [3] respactively.

Definition 1.1[7]. Let $A, B \in I^X$ and $x_\lambda \in F_p(X)$, where $F_p(X)$ denotes the set of all fuzzy points in X. Then :

(1) A is said to be *quasi-coincident with* B, denoted by AqB, if there exists an $x \in X$ such that $A(x) > B^c(x)$ or $A(x) + B(x) > 1$.

Also, we say that A and B are *quasi-coincident*(with each other) at x.

(2) x_λ is said to be *quasi-coincident with* A, denoted by $x_\lambda q A$, if $\lambda > A^c(x)$ or $\lambda + A(x) > 1$.

Definition 1.2[3]. A fts X is said To be :

(1) T_1, if for any pair of distinct points x_λ and y_μ,

(case1). When $x \neq y$, x_λ has an open neighborhood which is not q-coincident with y_μ and y_μ has an open neighborhood which is not q-coincident with x_λ.

(case2). When $x = y$, and $\lambda < \mu$(say), then there exists a q-neighborhood V of y_μ such that $x_\lambda \bar{q} V$.

(2) T_2 or *Hausdorff*, if for any two distinct points x_λ and y_μ,

(case1). When $x \neq y$, x_λ have open neighborhoods which are not q-coincident.

(case2). When $x = y$, and $\lambda < \mu$(say), then y_μ has an open q-neighborhood V and x_λ has open neighborhood U such that $V \bar{q} U$.

Result 1.A[3]. A fts X is T_1 if and only if every singleton set is closed in X.

Definition 1.3[4]. A collection of fuzzy sets $\mathcal{F}$ of a set X is said to form a *filter base*, if for any finite collection $\{U_i: i = 1, \cdots, n\}$ form $\mathcal{F}$, $\bigcap_{i=1}^{n} U_i \neq \emptyset$.

Definition 1.4[4]. A fuzzy set A in a fts X is said to be compact if for each filter base $\mathcal{F}$ such that every finite intersection of members of $\mathcal{F}$ is q-coincident with A, $\left(\bigcap_{B\in\mathcal{F}} \overline{B}\right) \cap A \neq \emptyset$.

Result 1.B[4]. Every fuzzy closed set of a compact fts is compact.

Result 1.C[4]. A compact subset of a T_2-fts is closed.

Notation 1.5[5]. Let X be a fts and let $A \in I^X$. Then :

(1) $I_0^X = \{ E : E \text{ is a nonempty fuzzy closed set in } X \}$.

(2) $I_0^A = \{ E \in I_0^X : E \subset A \}$.

Definition 1.6[5]. Let $(X, \mathfrak{T})$ be a fts. Then the *fuzzy Vietories topology* $\mathfrak{T}_v$ on I_0^X is generated by the collection of the forms $\langle U_1, \cdots, U_n \rangle_v$ with $U_1, \cdots, U_n$ fuzzy open sets in X, where $\langle U_1, \cdots, U_n \rangle_v = \{ E \in I_0^X : E \subset \bigcup_{i=1}^{n} U_i \text{ and } E\, q\, U_i \text{ for each } i = 1, \cdots n \}$.

The pair $(I_0^X, \mathfrak{T}_v)$ is called a *fuzzy hyperspace with fuzzy Vietories topology(fuzzy hyperspace*, in short).

Definition 1.7[6]. A mapping is said to be *fuzzy set-valued* if its values are fuzzy sets in a given set.

Result 1.D[6]. Let Y be a fts, I_0^X a fuzzy hyperspace and $F: Y \to I_0^X$ a fuzzy set-valued mapping. Then the following are equivalent :

(1) F is F-continuous

(2) For each fuzzy open(resp. closed) set A in X,

$$F^{-1}(I_0^A) = \{ y \in Y : F(y) \in I_0^A \} = \{ y \in Y : F(y) \subset A \}$$

is open(resp. closed) in Y.

(3) For each fuzzy closed (resp. open) set A in X,

$$Y - F^{-1}(I_0^{A^c}) = \{ y \in Y : F(y) \notin I_0^{A^c} \} = \{ y \in Y : F(y)\, q A \},$$

is closed(resp. open) in Y.

2. Definition and foundamental properties.

Notation. Let X be a fts and $E \in I^X$. Then :

(1) $[I_0^X] = \{ E \in I^X : E$ is closed in $X \}$. (2) $E^- = \{ A \in [I_0^X] : AqE \}$.

(3) $E^+ = \{ A \in [I_0^X] : A \subset E \}$.

(4) $\mathcal{F}_n = \{ E \in I_0^X : E$ has at most n elements $\}$.

Definition 2.1. Let X be a fts and let

$$\mathcal{S} = \{ V^- : V \text{ is open in } X \} \bigcup \{ (K^c)^+ : K \text{ is compact in } X \}$$

Then the *fuzzy Fell topology* T_f on $[I_0^X]$ has a subbase $\mathcal{S}$.

Theorem 2.2. The basic elements of T_f are three kinds of the forms :

Type 1. $\bigcap_{i=1}^{n} V_i^-$, where each V_i is open in X.

Type 2. $(K^c)^+$, where K is compact in X.

Type 3. $(\bigcap_{i=1}^{n} V_i^-) \bigcap (K^c)^+$.

Theorem 2.3. Let X be a Hausdorff fts. Then the fuzzy Vietories topology T_v on I_0^X is finer than the fuzzy Fell topology T_f on I_0^X.

Furthermore, if we replace fuzzy compact sets in the definition of the subbase for T_f by fuzzy closed sets, then T_f and T_v are equivalent. Hence if X is fuzzy compact Hausdorff, then $T_f = T_v$.

Theorem 2.4. Let X be a Hausdorff fts. Then :

(1) If $f: Z \to (I_0^X, T_v)$ is a F-continuous, fuzzy set-valued mapping, then $f: Z \to (I_0^X, T_f)$ is F-continuous.

(2) Let $f: (I_0^X, T_f)^n \to (I_0^X, T_f)$ be defined by $f(A_1, \cdots, A_n) = \bigcup_{i=1}^{n} A_i$. Then f is F-continuous.

Theorem 2.5. Let X be a fuzzy Hausdorff space. If $\mathcal{O}$ is open in (I_0^X, T_f), then $\bigcup \mathcal{O} = \bigcup \{ E : E \in \mathcal{O} \}$ is open in X.

3. Further properties

Theorem 3.1. Let X be a fuzzy Hausdorff space. Then :

(1) If $\mathscr{B}$ is a compact subset of (I_0^X, T_v), then it is a compact subset of (I_0^X, T_f).

(2) If $\mathscr{D}$ is dense in (I_0^X, T_v), then it is also dense in (I_0^X, T_f).

Theorem 3.2. Let X be a fts. Then :

(1) $\{ E \in I_0^X : E \subset A \}$ is closed in (I_0^X, T_f) if A is closed in X.

(2) $\{ E \in I_0^X : EqA \}$ is closed in (I_0^X, T_f) if A is compact in X and X is Hausdorff.

Let $\mathcal{F}_n(X) = \{E \in I_0^X : X$ has at most n elements with distinct support $\}$.
Then we have the following result :

Theorem 3.3. Let X be a fuzzy Hausdorff space. If $\mathcal{O}$ is open in $(\mathcal{F}_n(X), T_f)$. Then $\bigcup \mathcal{O}$ is open in X.

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