Fuzzy (r, s)-interiors and fuzzy (r, s)-closures

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Abstract

We introduce the concept of double fuzzy topological spaces as a generalization of intuitionistic fuzzy topological spaces and smooth topological spaces and then investigate some of their properties. Also we introduce the notions of fuzzy (r, s)-interiors and fuzzy (r, s)-closures in double fuzzy topological spaces.

1. Introduction

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues [5, 6, 7] introduced intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets.

In this paper, we introduce the concept of double fuzzy topological spaces as a generalization of intuitionistic fuzzy topological spaces and smooth topological spaces and then investigate some of their properties. Also we introduce the notions of fuzzy (r, s)-interiors and fuzzy (r, s)-closures in double fuzzy topological spaces.

2. Preliminaries

Let X be a nonempty set. An *intuitionistic fuzzy set* A is an ordered pair

$$A = (\mu_A, \gamma_A)$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively, and $\mu_A + \gamma_A \leq \tilde{1}$.

Obviously every fuzzy set μ of X is an intuitionistic fuzzy set of the form $(\mu, \tilde{1} - \mu)$.

Definition 2.1. [1] Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy sets on X. Then

(1) $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\gamma_A \geq \gamma_B$.

(2) $A = B$ iff $A \subseteq B$ and $A \supseteq B$.

(3) $A^c = (\gamma_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.

(6) $0_\sim = (\tilde{0}, \tilde{1})$ and $1_\sim = (\tilde{1}, \tilde{0})$.

Definition 2.2. [6] An *intuitionistic fuzzy* *topology* on X is a family $\Im$ of intuitionistic fuzzy sets in X which satisfies the following properties:

(1) $0_{\sim}, 1_{\sim} \in \Im$.

(2) If $A_1, A_2 \in \Im$ then $A_1 \cap A_2 \in \Im$.

(3) If $A_i \in \Im$ for all i, then $\bigcup A_i \in \Im$.

The pair $(X, \Im)$ is called an *intuitionistic fuzzy topological space.*

Let $(X, \Im)$ be an intuitionistic fuzzy topological space. Then members of $\Im$ are called *intuitionistic fuzzy open sets* of X and their complements *intuitionistic fuzzy closed sets*.

3. Double fuzzy topological spaces

Let $I(X)$ be a family of all intuitionistic fuzzy sets of X.

Definition 3.1. Let X be a nonempty set. A *double fuzzy topology* $\Im = (\tau, \tau_c)$ on X is two maps $\tau : I(X) \to I$ and $\tau_c : I(X) \to I$ which satisfy the following properties:

(1) $\tau(A) + \tau_c(A) \leq 1$ for any $A \in I(X)$.

(2) $\tau(0_{\sim}) = \tau(1_{\sim}) = 1$ and $\tau_c(0_{\sim}) = \tau_c(1_{\sim}) = 0$.

(3) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau_c(A \cap B) \leq \tau_c(A) \vee \tau_c(B)$.

(4) $\tau(\bigcup A_i) \geq \bigwedge \tau(A_i)$ and $\tau_c(\bigcup A_i) \leq \bigvee \tau_c(A_i)$.

The $(X, \Im) = (X, \tau, \tau_c)$ is called a *double fuzzy topological space*. And, we call τ a *gradation of openness* and τ_c a *gradation of nonopenness.*

Definition 3.2. Let X be a nonempty set. A *double fuzzy family of closed sets* $\Omega = (\omega, \omega_c)$ on X is two maps $\omega : I(X) \to I$ and $\omega_c : I(X) \to I$ which satisfy the following properties:

(1) $\omega(A) + \omega_c(A) \leq 1$ for any $A \in I(X)$.

(2) $\omega(0_{\sim}) = \omega(1_{\sim}) = 1$ and $\omega_c(0_{\sim}) = \omega_c(1_{\sim}) = 0$.

(3) $\omega(A \cup B) \geq \omega(A) \wedge \omega(B)$ and $\omega_c(A \cup B) \leq \omega_c(A) \vee \omega_c(B)$.

(4) $\omega(\bigcap A_i) \geq \bigwedge \omega(A_i)$ and $\omega_c(\bigcap A_i) \leq \bigvee \omega_c(A_i)$.

In this case, we call ω a *gradation of closedness* and ω_c a *gradation of nonclosedness.*

Proposition 3.3. Let $\Im = (\tau, \tau_c)$ be a double fuzzy topology on X and $\Omega_{\Im} = (\omega_{\tau}, \omega_{\tau_c})$ defined by

$$\omega_{\tau}(A) = \tau(A^c) \text{ and } \omega_{\tau_c}(A) = \tau_c(A^c).$$

Then $\Omega_{\Im}$ is a double fuzzy family of closed sets on X.

Proposition 3.4. Let $\Omega = (\omega, \omega_c)$ be a double fuzzy family of closed sets on X and $\Im_{\Omega} = (\tau_{\omega}, \tau_{\omega_c})$ defined by

$$\tau_{\omega}(A) = \omega(A^c) \text{ and } \tau_{\omega_c}(A) = \omega_c(A^c).$$

Then $\Im_{\Omega}$ is a double fuzzy topology on X.

Corollary 3.5. Let $\Im = (\tau, \tau_c)$ be a double fuzzy topology and $\Omega=(\omega, \omega_c)$ a double fuzzy family of closed sets. Then

$$\mathfrak{T}_{\Omega_{\mathfrak{T}}}=\mathfrak{T} \text{ and } \Omega_{\mathfrak{T}_\Omega}=\Omega.$$

Proposition 3.6. Let (X, τ, τ_c) be a double fuzzy topological space. Then for each $r\in I$,

$$\tau_r=\{A\in I(X) \mid \tau(A)\geq r\}$$

is an intuitionistic fuzzy topology on X. Moreover $\tau_{r_1}\supseteq\tau_{r_2}$ if $r_1\leq r_2$.

Proposition 3.7. Let (X, τ, τ_c) be a double fuzzy topological space. Then for each $s\in I$,

$$\tau_c{}^s=\{A\in I(X) \mid \tau_c(A)\leq s\}$$

is an intuitionistic fuzzy topology on X. Moreover $\tau_c{}^{s_1}\subseteq\tau_c{}^{s_2}$ if $s_1\leq s_2$.

Corollary 3.8. Let $\mathfrak{T}=(\tau, \tau_c)$ be a double fuzzy topology on X. Then for each $r, s\in I$,

$$\mathfrak{T}_r^s=\{A\in I(X) \mid \tau(A)\geq r \text{ and } \tau_c(A)\leq s\}$$

is an intuitionistic fuzzy topology on X and $\mathfrak{T}_r^s=\tau_r\cap\tau_c{}^s$. Moreover $\mathfrak{T}_{r_2}^{s_1}\subseteq\mathfrak{T}_{r_1}^{s_2}$ if $r_1\leq r_2$ and $s_1\leq s_2$.

4. Fuzzy (r, s)-interiors and fuzzy (r, s)-closures

Let $I\oplus I=\{(r, s) \mid r, s\in I \text{ and } r+s\leq 1\}$.

Definition 4.1. Let A be an intuitionistic fuzzy set of a double fuzzy topological space (X, τ, τ_c) and $(r, s)\in I\oplus I$. Then A is called;

(1) a *fuzzy* (r, s)*-open* set if $\tau(A)\geq r$ and $\tau_c(A)\leq s$,

(2) a *fuzzy* (r, s)*-closed* set if $\tau(A^c)\geq r$ and $\tau_c(A^c)\leq s$.

Definition 4.2. Let (X, τ, τ_c) be a double fuzzy topological space. For each $(r, s)\in I\oplus I$ and for each $A\in I(X)$, the *fuzzy* (r, s)*-interior* is defined by

$$\mathrm{Int}(A, r, s)=\bigcup\{B\in I(X) \mid A\supseteq B \text{ and } B \text{ is a fuzzy } (r, s)\text{-open set}\}$$

and the *fuzzy* (r, s)*-closure* is defined by

$$\mathrm{Cl}(A, r, s)=\bigcap\{B\in I(X) \mid A\subseteq B \text{ and } B \text{ is a fuzzy } (r, s)\text{-closed set}\}.$$

The operators $\mathrm{Int}: I(X)\times I\oplus I\to I(X)$ and $\mathrm{Cl}: I(X)\times I\oplus I\to I(X)$ are called the *fuzzy interior operator* and the *fuzzy closure operator* in (X, τ, τ_c), respectively.

Obviously, $\mathrm{Int}(A, r, s)$ is the greatest fuzzy (r, s)-open set which is contained in A and $\mathrm{Cl}(A, r, s)$ is the smallest fuzzy (r, s)-closed set which contains A. Also, $\mathrm{Int}(A, r, s)=A$ for any fuzzy (r, s)-fuzzy open set A and $\mathrm{Cl}(A, r, s)=A$ for any fuzzy (r, s)-closed set A. Moreover, we have the following results.

Proposition 4.3. Let (X, τ, τ_c) be a double fuzzy topological space and let $\mathrm{Int}: I(X)\times I\oplus I\to I(X)$ the fuzzy interior operator and $\mathrm{Cl}: I(X)\times I\oplus I\to I(X)$ the fuzzy closure operator in (X, τ, τ_c). Then for any $A, B \in I(X)$ and $(r,s) \in I \oplus I$,

(1) $\mathrm{Int}(0_\sim, r, s) = 0_\sim$, $\mathrm{Int}(1_\sim, r, s) = 1_\sim$.

(2) $\mathrm{Int}(A, r, s) \subseteq A$.

(3) $\mathrm{Int}(A, r_1, s_1) \supseteq \mathrm{Int}(A, r_2, s_2)$ if $r_1 \le r_2$ and $s_1 \ge s_2$.

(4) $\mathrm{Int}(A \cap B, r, s) = \mathrm{Int}(A, r, s) \cap \mathrm{Int}(B, r, s)$.

(5) $\mathrm{Int}(\mathrm{Int}(A, r, s), r, s) = \mathrm{Int}(A, r, s)$.

(6) $\mathrm{Cl}(0_\sim, r, s) = 0_\sim$, $\mathrm{Cl}(1_\sim, r, s) = 1_\sim$.

(7) $\mathrm{Cl}(A, r, s) \supseteq A$.

(8) $\mathrm{Cl}(A, r_1, s_1) \subseteq \mathrm{Cl}(A, r_2, s_2)$ if $r_1 \le r_2$ and $s_1 \ge s_2$.

(9) $\mathrm{Cl}(A \cup B, r, s) = \mathrm{Cl}(A, r, s) \cup \mathrm{Cl}(B, r, s)$.

(10) $\mathrm{Cl}(\mathrm{Cl}(A, r, s), r, s) = \mathrm{Cl}(A, r, s)$.

Proposition 4.4. Let $\mathrm{Int} : I(X) \times I \oplus I \to I(X)$ be a map satisfying (1)-(5) of Theorem 4.3. Let $\tau : I(X) \to I$ and $\tau_c : I(X) \to I$ be maps defined by

$$\tau(A) = \bigvee \{ r \in I \mid \mathrm{Int}(A, r, s) = A \}$$

and

$$\tau_c(A) = \bigwedge \{ s \in I \mid \mathrm{Int}(A, r, s) = A \}.$$

Then $\mathfrak{I} = (\tau, \tau_c)$ is a double fuzzy topology on X.

Proposition 4.5. Let $\mathrm{Cl} : I(X) \times I \oplus I \to I(X)$ be a map satisfying (6)-(10) of Theorem 4.3. Let $\omega : I(X) \to I$ and $\omega_c : I(X) \to I$ be maps defined by

$$\omega(A) = \bigvee \{ r \in I \mid \mathrm{Cl}(A, r, s) = A \}$$

and

$$\omega_c(A) = \bigwedge \{ s \in I \mid \mathrm{Cl}(A, r, s) = A \}.$$

Then $\Omega = (\omega, \omega_c)$ is a double fuzzy family of closed sets on X.

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