FUZZY STRONGLY r-SEMICONTINUOUS NEIGHBORHOODS

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Abstract. In this thesis, we introduce and investigate the notions of a fuzzy strongly r-semineighborhood and a fuzzy strongly r-quasi-semineighborhood in fuzzy topological spaces which are generalizations of a fuzzy strongly semineighborhood and a fuzzy strongly quasi-semineighborhood, respectively.

1. Introduction and Preliminaries

Definition 1.1. ([6]) Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is called

(1) a *fuzzy r-open set* of X if $\mathcal{T}(\mu) \geq r$,
(2) a *fuzzy r-closed set* of X if $\mathcal{T}(\mu^c) \geq r$.

Definition 1.2. ([3,6]) Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-closure* is defined by

$$\mathrm{cl}(\mu, r) = \bigwedge\{\rho \in I^X : \mu \leq \rho, \mathcal{T}(\rho^c) \geq r\}.$$

and the *fuzzy r-interior* is defined by

$$\mathrm{int}(\mu, r) = \bigvee\{\rho \in I^X : \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

Definition 1.3. ([6,7]) Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathrm{cl}(\rho, r)$,
(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{int}(\rho, r) \leq \mu \leq \rho$,
(3) *fuzzy r-preopen* if $\mu \leq \mathrm{int}(\mathrm{cl}(\mu, r), r)$,
(4) *fuzzy r-preclosed* if $\mathrm{cl}(\mathrm{int}(\mu, r), r) \leq \mu$.

Definition 1.4. ([5]) Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy strongly r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathrm{int}(\mathrm{cl}(\rho, r), r)$,
(2) *fuzzy strongly r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{cl}(\mathrm{int}(\rho, r), r) \leq \mu \leq \rho$.

Theorem 1.5. ([5])

(1) *Any union of fuzzy strongly r-semiopen sets is fuzzy strongly r-semiopen.*
(2) *Any intersection of fuzzy strongly r-semiclosed sets is fuzzy strongly r-semiclosed.*

Definition 1.6. ([5]) Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy strongly r-semiclosure* is defined by

$$\mathrm{sscl}(\mu, r) = \bigwedge\{\rho \in I^X : \mu \leq \rho, \rho \text{ is fuzzy strongly } r\text{-semiclosed}\},$$

and the *fuzzy strongly r-semiinterior* is defined by

$$\mathrm{ssint}(\mu, r) = \bigvee\{\rho \in I^X : \mu \geq \rho, \rho \text{ is fuzzy strongly } r\text{-semiopen}\}.$$

Definition 1.7. ([4,7]) Let x_α be a fuzzy point of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then a fuzzy set μ of X is called

(1) a *fuzzy r-neighborhood* (*fuzzy r-semineighborhood, fuzzy r-preneighborhood*, respectively) of x_α if there is a fuzzy r-open(fuzzy r-semiopen, fuzzy r-preopen, respectively) set ρ in X such that $x_\alpha \in \rho \leq \mu$,

(2) a *fuzzy r-quasi-neighborhood* (*fuzzy r-quasi-semineighborhood, fuzzy r-quasi-preneighborhood*, respectively) of x_α it there is a fuzzy r-open(fuzzy r-semiopen, fuzzy r-preopen, respectively) set ρ in X such that $x_\alpha \mathrm{q} \rho \leq \mu$,

2. Fuzzy strongly r-semineighborhoods

We are going to define the concepts of a fuzzy strongly r-semineighborhood and a fuzzy strongly r-quasi-semineighborhood in a fuzzy topological space.

Definition 2.1. Let x_α be a fuzzy point of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then a fuzzy set μ of X is called

(1) a *fuzzy strongly r-semineighborhood* of x_α if there is a fuzzy strongly r-semiopen set ρ in X such that $x_\alpha \in \rho \leq \mu$,

(2) a *fuzzy strongly r-quasi-semineighborhood* of x_α if there is a fuzzy strongly r-semiopen set ρ in X such that $x_\alpha \mathrm{q} \rho \leq \mu$.

Clearly, if μ is a fuzzy strongly r-semineighborhood (strongly r-quasi-semineighborhood) of x_α and $r \geq t$, then μ is also a fuzzy strongly t-semineighborhood (strongly t-quasi-semineighborhood) of x_α.

Theorem 2.2. *Let $(X, \mathcal{T})$ be a fuzzy topological space and $r \in I_0$. Then a fuzzy set μ of X is fuzzy strongly r-semiopen if and only if μ is a fuzzy strongly r-semineighborhood of x_α for every fuzzy point $x_\alpha \in \mu$.*

Theorem 2.3. *Let $(X, \mathcal{T})$ be a fuzzy topological space and $r \in I_0$. Then a fuzzy set μ of X is fuzzy strongly r-semiopen if and only if μ is a fuzzy strongly r-quasi-semineighborhood of x_α for every fuzzy point $x_\alpha \mathrm{q} \mu$.*

Theorem 2.4. *Let x_α be a fuzzy point in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then $x_\alpha \in \mathrm{sscl}(\mu, r)$ if and only if $\rho \mathrm{q} \mu$ for all fuzzy strongly r-quasi-semineighborhood ρ of x_α.*

Theorem 2.5. *Let x_α be a fuzzy point in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then $x_\alpha \in \mathrm{ssint}(\mu, r)$ if and only if there is a fuzzy strongly r-semineighborhood ρ of x_α such that $\rho \leq \mu$.*

Remark 2.6.

(1) Every fuzzy r-neighborhood (r-quasi-neighborhood) of x_α is also a fuzzy strongly r-semineighborhood (strongly r-quasi-semineighborhood) of x_α.

(2) Every fuzzy strongly r-semineighborhood (strongly r-quasi-semineighborhood) of x_α is also a fuzzy r-semineighborhood (r-quasi-semineighborhood) of x_α.

(3) Every fuzzy strongly r-semineighborhood (strongly r-quasi-semineighborhood) of x_α is also a fuzzy r-preneighborhood (r-quasi-preneighborhood) of x_α.

Following examples show that their converses need not be true in general.

Example 2.7. Let $X = \{a, b\}$ and μ_1 and μ_2 be fuzzy sets of X defined by

$$\mu_1(a) = \frac{3}{5}, \qquad \mu_1(b) = \frac{1}{10};$$

and

$$\mu_2(a) = \frac{7}{10}, \qquad \mu_2(b) = \frac{9}{10}.$$

Define $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}$ is a fuzzy topology on X. Let $x = b$ and $\alpha = \frac{1}{5}$. Then $x_\alpha \in \mu_2$ and μ_2 is fuzzy strongly $\frac{1}{2}$-semiopen. Thus μ_2 is a fuzzy strongly $\frac{1}{2}$-semineighborhood but not fuzzy $\frac{1}{2}$-neighborhood. Also μ_2 is a fuzzy strongly $\frac{1}{2}$-quasi-semineighborhood of x_α which is not a fuzzy $\frac{1}{2}$-quasi-neighborhood of x_α.

Example 2.8. Let $X = \{a, b\}$ and μ_1 and μ_2 be fuzzy sets of X defined by

$$\mu_1(a) = \frac{1}{2}, \qquad \mu_1(b) = \frac{2}{5};$$

and

$$\mu_2(a) = \frac{1}{2}, \qquad \mu_2(b) = \frac{3}{5}.$$

Define $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}$ is a fuzzy topology on X. Let $x = b$ and $\alpha = \frac{1}{2}$. Then $x_\alpha \in \mu_2$ and μ_2 is fuzzy $\frac{1}{2}$-semiopen. Thus μ_2 is a fuzzy $\frac{1}{2}$-semineighborhood but not fuzzy strongly $\frac{1}{2}$-semineighborhood. Also μ_2 is a fuzzy $\frac{1}{2}$-quasi-semineighborhood of x_α which is not a fuzzy strongly $\frac{1}{2}$-quasi-semineighborhood of x_α.

Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$, an r-cut

$$\mathcal{T}_r = \{\mu \in I^X : \mathcal{T}(\mu) \geq r\}$$

is a Chang's fuzzy topology on X.

Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. A fuzzy topology $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

The next two theorems show that a fuzzy strongly semineighborhood[11] is a special case of a fuzzy strongly r-semineighborhood.

Theorem 2.10. *Let x_α be a fuzzy point of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then a fuzzy set μ is a fuzzy strongly r-semineighborhood (strongly r-quasi-semineighborhood) of x_α in $(X, \mathcal{T})$ if and only if μ is a fuzzy strongly semineighborhood (strongly quasi-semineighborhood) of x_α in $(X, \mathcal{T}_r)$.*

Theorem 2.11. *Let x_α be a fuzzy point of a Chang's fuzzy topological space (X,T) and $r \in I_0$. Then a fuzzy set μ is a fuzzy strongly semineighborhood (strongly quasi-semineighborhood) of x_α in (X,T) if and only if μ is a fuzzy strongly r-semineighborhood (strongly r-quasi-semineighborhood) of x_α in (X,T^r).*

The *product fuzzy set* $\mu \times \rho$ of a fuzzy set μ of X and a fuzzy set ρ of Y is defined by

$$(\mu \times \rho)(x,y) = \mu(x) \wedge \rho(y)$$

for all $(x,y) \in X \times Y$.

Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be fuzzy topological spaces and $r \in I_0$. Then X is *r-product related* to Y if any fuzzy set μ of X and any fuzzy set ρ of Y,

$$\mathrm{cl}(\mu \times \rho, r) = \mathrm{cl}(\mu, r) \times \mathrm{cl}(\rho, r).$$

Let $\{(X_i, \mathcal{T}_i)\}_{i\in J}$ be a family of fuzzy topological spaces. Let $X = \prod X_i$ and $p_i : X \to X_i, i \in J$, denote the projection map. Let $(\mathcal{T}_i)_r$ denote the Chang's fuzzy topology on X_i for $i \in J$, $r \in I_0$. Let

$$\prod(\mathcal{T}_i)_r = \sup_{i\in J} p_i^{-1}((\mathcal{T}_i)_r)$$

be the Chang's fuzzy topology generated by $\{p_i^{-1}((\mathcal{T}_i)_r)\}_{i\in J}$ as a subbase. Let $\mathcal{T}$ be the fuzzy topology generated by $\{\prod(\mathcal{T}_i)_r\}_{0<r\leq 1}$. That is

$$\mathcal{T}(\mu) = \bigvee\{r \in I_0 : \mu \in \prod(\mathcal{T}_i)_r\}.$$

Then $\mathcal{T}$ is called the *product fuzzy topology* on X and denoted by $\prod \mathcal{T}_i$.

Lemma 2.12. *Let $r \in I_0$ and a fuzzy topological space $(X,\mathcal{T})$ be r-product related to a fuzzy topological space $(Y,\mathcal{U})$. Then for any fuzzy set μ of X and any fuzzy set ρ of Y, $\mathrm{int}(\mu \times \rho, r) = \mathrm{int}(\mu, r) \times \mathrm{int}(\rho, r)$.*

Theorem 2.13. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be fuzzy topological spaces and $r \in I_0$. If X is r-product related to Y, then the product $\mu \times \rho$ of a fuzzy strongly r-semiopen (strongly r-semiclosed) set μ in X and a fuzzy strongly r-semiopen (strongly r-semiclosed) set ρ in Y is fuzzy strongly r-semiopen (strongly r-semiclosed) in the product fuzzy topological space $X \times Y$.*

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