OPERATIONS ON FUZZY TOPOLOGICAL SPACES

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ABSTRACT

In this paper we introduce the notion of fuzzy γ-open sets by using an operation γ on fuzzy topological space (X, τ) and investigate the related fuzzy topological properties of the associated fuzzy topology τ_γ and τ. And γ-T_i (i=0,1,2) separation axioms are defined in fuzzy topological spaces and the validity of some results analogous to those in fuzzy T_i spaces due to Ganguly and Saha [2] are examined.

1. Introduction

Using the concept of a fuzzy set, Chang [1] introduced a fuzzy topological space (for short, fts). Since then, many authors [2-7] have contributed to the development of this theory. Ganguly and Saha [2] introduced fuzzy T_i (i=0,1,2) spaces and investigated their properties.

In this paper, we introduce the concept of an operation γ on a fts (X, τ) and use this concept to define fuzzy γ-open sets and investigate the related fuzzy topological properties of the associated fuzzy topology τ_γ and τ. Also we define fuzzy γ-closure and τ_γ-closure and study their relation and properties. Finally, we introduce the notions of fuzzy γ-T_i (i=0,1,2) spaces and characterize fuzzy γ-T_i by the notion of fuzzy γ-closed or fuzzy γ-open sets.

A fuzzy point in X with support $x \in X$ and value α ($0 < \alpha \leq 1$) is denoted by x_α. For a fuzzy set A of X, the notations Int(A), Cl(A) and $1-A$ will respectively stand for the fuzzy interior, fuzzy closure and complement of A. By 0_X and 1_X we will mean the constant fuzzy sets taking on respectively the values 0 and 1 on X.

A fuzzy point x_α is quasi-coincident (in short, q-coincident) with a fuzzy set A, denoted by $x_\alpha qA$, if $\alpha + A(x) > 1$. A fuzzy set A is q-coincident with a fuzzy set B, denoted by AqB, if there exists $x \in X$ such that $A(x)+B(x)>1$. If A is not q-coincident with B, then we write $A \not{q} B$.

A fuzzy set A in a fts X is said to be a fuzzy q-nbd of a fuzzy point x_α in X if there exists a fuzzy open set B such that $x_\alpha qB \leq A$.

2. Fuzzy γ-open sets

In this section we define the notion of an operation γ and fuzzy γ-open sets by using an operation γ on fts (X, τ)

Definition 2.1. Let (X, τ) be a fts. An operation γ on fuzzy topology τ is a mapping from τ into fuzzy set I^X of X such that $V \leq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma: \tau \to I^X$.

The operators defined by $\gamma(G)$=Int(G), $\gamma(G)$ =Cl(G) and $\gamma(G)$=Int(Cl(G)) are examples of the operation γ.

Definition 2.2. A fuzzy set A of a fts (X, τ) is called a fuzzy γ-open set of (X, τ) if, for each fuzzy point $x_\alpha \in A$, there exists a fuzzy open set U such that $x_\alpha \in U$ and $U^\gamma \leq A$. τ_γ will be denoted the set all fuzzy γ-open sets. A fuzzy set B of (X, τ) is said to be fuzzy γ-closed in (X, τ) if $1-B$ is fuzzy γ-open.

Proposition 2.3. $\tau_\gamma \subseteq \tau$.

Definition 2.4. Let γ be an operation on τ. Then γ is called:
(a) regular if for every fuzzy open neighborhood (in short, fo-nbd) U and V of each fuzzy point $x_\alpha \in X$, there exists a fo-nbd W of fuzzy point x_α such that $W^\gamma \leq U^\gamma \wedge V^\gamma$;

(b) open if for every fo-nbd U of each fuzzy point $x_\alpha \in X$, there exists a fuzzy γ-open set S such that $x_\alpha \in S$ and $S \leq U^\gamma$.

Example 2.5. Let X={a, b, c} and $\tau=\{0_X, A_1, A_2, 1_X\}$, where

$A_1(a)=A_1(b)=1, A_1(c)=0,$
$A_2(a)=A_2(c)=0, A_2(b)=0.5.$

Let $\gamma: \tau \rightarrow I^X$ be an operation defined by $\gamma(B)$=Cl(B), and let $\delta: \tau \rightarrow I^X$ be an operation defined by $\delta(B)$=Int(B). Then we have $\tau_\gamma=\{0_X, 1_X\}$ and $\tau_\delta=\tau$. It is easy to see that γ is regular but it is not open on (X, τ), and δ is regular and open on (X, τ).

Example 2.6. Let X={a, b, c} and $\tau=\{0_X, A_1, A_2, A_3, A_4, 1_X\}$, where

$A_1(a)=A_1(b)=A_1(c)=0.7,$
$A_2(a)=A_2(c)=0.7, A_2(b)=0.3$,
$A_3(a)=A_3(c)=0.3, A_3(b)=0.7$ and
$A_4(a)=A_4(b)=A_4(c)=0.3.$

Let $\gamma: \tau \rightarrow I^X$ be an operation defined by

$$\gamma(A)=A^\gamma=\begin{cases} 1_X & \text{if } A=A_4 \\ \text{Cl}(A) & \text{otherwise.} \end{cases}$$

Then the operation $\gamma: \tau \rightarrow I^X$ is open but not regular on τ.

Proposition 2.7. Let $\gamma: \tau \rightarrow I^X$ be a regular operation on τ.
(a) If A and B are fuzzy γ-open, then $A \wedge B$ is fuzzy γ-open.
(b) τ_γ is a fuzzy topology.

Remark 2.8. In general, τ_γ is supra fuzzy topology but not fuzzy topology on X.

Example 2.9. Let (X, τ) be a fts in Example 2.6. Then A_2 and A_3 are fuzzy γ-open sets but $A_2 \wedge A_3$ is not fuzzy γ-open. τ_γ is supra fuzzy topology but not fuzzy topology.

Now, we define the notion of γ-closure of a fuzzy set of a fts (X, τ) as follows:

Definition 2.10. (a) A fuzzy point $x_\alpha \in X$ is in the γ-closure of a fuzzy set A of fts X if $U^\gamma qA$ for each fo-q-nbd U of x_α. The γ-closure of a fuzzy set A is denoted by $\text{Cl}_\gamma(A)$.
(b) $\tau_\gamma\text{-Cl}(A)=\bigwedge\{F : F \geq A,\ 1-F \in \tau_\gamma\}$.

Proposition 2.11. For each fuzzy point x_α in X, $x_\alpha \in \tau_\gamma\text{-Cl}(A)$ if and only if VqA for any $V \in \tau_\gamma$ such that $x_\alpha \in V$.

Remark 2.12. Let $\gamma: \tau \rightarrow I^X$ be an operation on τ and A be a fuzzy set of X.
(a) $A \leq \text{Cl}(A) \leq \text{Cl}_\gamma(A) \leq \tau_\gamma\text{-Cl}(A)$.
(b) The fuzzy set Cl(A) is fuzzy closed in (X, τ).
(c) If γ is open, then $\text{Cl}_\gamma(A)=\tau_\gamma\text{-Cl}(A)$ and $\text{Cl}_\gamma(A)$ is fuzzy γ-closed in fts (X, τ) (i.e. $\text{Cl}_\gamma(\text{Cl}_\gamma(A))=\text{Cl}_\gamma(A)$).

In the Remark 2.12, $\text{Cl}_\gamma(A)$ is proper subset of $\tau_\gamma\text{-Cl}(A)$ as the following example.

Example 2,13. Let X={a, b, c} and $\tau=\{0_X, A_1, A_2, A_3, 1_X\}$, where

$A_1(a)=1, A_1(b)=A_1(c)=0,$
$A_2(a)=A_2(c)=0, A_2(b)=1$, and

$A_3(a) = A_3(b) = 1, A_3(b) = 0.$

Let $\gamma: \tau \rightarrow I^X$ be an operation defined by $\gamma(B)$=Cl(B). Take B=A_1. Then $\mathrm{Cl}_\gamma(A_1)$ is proper subset of τ_γ-Cl(A_1).

Theorem 2.14. For a fuzzy set A of (X, τ), the following are equivalent:
(a) A is fuzzy γ-open in (X, τ).
(b) $\mathrm{Cl}_\gamma(X-A) = X-A$.
(c) τ_γ-Cl(X-A) = X-A.

Remark 2.15. $\mathrm{Cl}_\gamma(\mathrm{Cl}_\gamma(A)) \neq \mathrm{Cl}_\gamma(A)$. (See Example 2.13.)

Lemma 2.16. If γ is regular operation, then $\mathrm{Cl}_\gamma(A \vee B) = \mathrm{Cl}_\gamma(A) \vee \mathrm{Cl}_\gamma(B)$.

From the Remark 2.12 and Lemma 2.16, we notice:

Corollary 2.17. If γ is regular and open on (X, τ), then the operation Cl_γ satisfies the Kuratowski closure axiom. i.e., τ_γ={$A \in I^X$: $\mathrm{Cl}_\gamma(1-A) = 1-A$} is fuzzy topology on X.

3. Fuzzy $\gamma - T_i$ spaces (i=0,1,2)

In this section we investigate general operator approaches of fuzzy T_i (i=0,1,2) spaces due to Ganguly and Saha [2].

Definition 3.1. A fts (X, τ) is called fuzzy γ-T_0 if for any of distinct points x_α and y_β:

Case I. When $x \neq y$, x_α has a fo-nbd U such that $y_\beta \not\mathrel{q} U^\gamma$, or y_β has a fo-nbd V such that $x_\alpha \not\mathrel{q} U^\gamma$.

Case II. When $x = y$ and $\alpha < \beta$ (say), there exists a fo-q-nbd U of x_α such that $y_\beta \not\mathrel{q} U^\gamma$.

Definition 3.2. A fts (X, τ) is called fuzzy γ-T_1 if for any of distinct points x_α and y_β:

Case I. When $x \neq y$, x_α has a fo-nbd U and y_β has a fo-nbd V such that $x_\alpha \not\mathrel{q} V^\gamma$ and $y_\beta \not\mathrel{q} U^\gamma$.

Case II. When $x = y$ and $\alpha < \beta$(say), there exists a fo-q-nbd V of y_β such that $x_\alpha \not\mathrel{q} V^\gamma$.

Definition 3.3. A fts (X, τ) is called fuzzy γ-T_2 if for any of distinct points x_α and y_β:

Case I. When $x \neq y$, x_α and y_β have fo-nbds U and V such that $U^\gamma \not\mathrel{q} V^\gamma$.

Case II. When $x = y$ and $\alpha < \beta$(say), x_α has a fo-q-nbd U and y_β has a fo-q-nbd V such that $U^\gamma \not\mathrel{q} V^\gamma$.

Remark 3.4. From above Definition 4.1-4.3 and Definition 3.1, 3.3 and 3.5 in [2], we obtain the following diagram:

$$\begin{array}{ccccc} \text{fuzzy } \gamma\text{-}T_2 & \rightarrow & \text{fuzzy } \gamma\text{-}T_1 & \rightarrow & \text{fuzzy } \gamma\text{-}T_0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{fuzzy } T_2 & \rightarrow & \text{fuzzy } T_1 & \rightarrow & \text{fuzzy } T_0 \end{array}$$

Theorem 3.5. A fts (X, τ) is fuzzy γ-T_0 if and only if for any pair of distinct fuzzy points x_α and y_β, either $x_\alpha \notin \mathrm{Cl}_\gamma(y_\beta)$ or $y_\beta \notin \mathrm{Cl}_\gamma(x_\alpha)$.

Theorem 3.6. A fts (X, τ) is fuzzy γ-T_1 if and only if every singleton fuzzy set is fuzzy γ-closed in (X, τ).

Throughout the rest of this section let (X, τ) and (Y, σ) be fuzzy topological spaces, and let $\gamma: \tau \rightarrow I^X$ and $\beta: \sigma \rightarrow I^X$ be operations on τ and σ, respectively.

Definition 3.7. A mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be fuzzy (γ, β)-continuous if for each fuzzy point x_α in X and each fo-q-nbd V of $f(x_\alpha)$, there exists a fo-q-nbd U of x_α such that $f(U^\gamma) \leq V^\beta$.

Proposition 3.8. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a mapping.
(a) f is fuzzy (γ, β)-continuous.
(b) $f(\mathrm{Cl}_\gamma(A)) \leq \mathrm{Cl}_\beta(f(A))$ hold for every fuzzy set A of X.
(c) For any fuzzy β-closed set B of fts Y, $f^{-1}(B)$ is fuzzy γ-closed in fts X (i.e., for

any $U \in \sigma_\beta$, $f^{-1}(U) \in \tau_\gamma$).
Then (a) $\Rightarrow$(b) $\Rightarrow$(c) hold.

Remark 3.9. (a) In Proposition 3.8, if Y is fuzzy β-regular, then (c) implies (a) and hence (a), (b) and (c) are equivalent to each other.

(b) In Proposition 3.8, if β is an open operation, then (b) implies (a) and hence (a) and (b) are equivalent to each other.

Recall that a mapping $f: (X, \tau) \to (Y, \sigma)$ is said to be fuzzy continuous[3] if for each fuzzy point x_α in X and each fo-q-nbd V of $f(x_\alpha)$, there exists a fo-q-nbd U of x_α such that $f(U) \leq V$.

The following two examples show that fuzzy (γ, β)-continuity and fuzzy continuity are independent concept.

Example 3.10. Let $X = Y = [0, 1]$ and $\tau = \sigma = \{1_X, 0_X, A\}$, where

$$A(x) = \begin{cases} 1/3 & \text{if } x = 0 \\ A(x) = 0 & \text{if } x \neq 0, \end{cases}$$

for each $x \in [0, 1]$.

Consider the identity mapping $f: (X, \tau) \to (Y, \sigma)$ and define $\gamma: \tau \to I^X$ by $U^\gamma = \mathrm{Cl}(U)$ for any $U \in \tau$ and $\beta: \sigma \to I^X$ by $G^\beta = \mathrm{Int}(\mathrm{Cl}(G))$ for any $G \in \sigma$. Then f is fuzzy continuous mapping but not fuzzy (γ, β) -continuous

Example 3.11. Let X be non-empty set, and let $f: X \to X$ be a identity mapping. Let a be fixed element of X, and σ the fuzzy topology on X given by $\sigma = \{1_X, 0_X, A\}$, where

$$A(x) = \begin{cases} \alpha\ (> \frac{1}{2}) & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$$

Let τ be any fuzzy topology on X such that $f: (X, \tau) \to (X, \sigma)$ is not fuzzy continuous (obviously such fuzzy topology exist). Define $\gamma: \tau \to I^X$ by $U^\gamma = \mathrm{Cl}(U)$ for any $U \in \tau$ and $\beta: \sigma \to I^X$ by $G^\beta = \mathrm{Int}(\mathrm{Cl}(G))$ for any $G \in \sigma$. Then f is fuzzy (γ, β)-continuous mapping but not fuzzy continuous.

Proposition 3.12. Let $f: (X, \tau) \to (Y, \sigma)$ be a fuzzy (γ, β)-continuous injection. If fts Y is fuzzy β-T_1 (resp. fuzzy β-T_2), then fts X is fuzzy γ-T_1 (resp. fuzzy γ-T_2).

Theorem 3.13. (a) Suppose that γ is regular. If (X, τ_γ) is a fuzzy T_2 space, then (X, τ) is a fuzzy γ-T_2 space.

(b) Suppose that γ is regular and open. If (X, τ) is a fuzzy γ-T_2 space, Then (X, τ_γ) is a fuzzy T_2 space.

References

[1] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182-190.

[2] S. Ganguly and S. Saha, On separation axioms and T_i-Fuzzy continuity, Fuzzy Sets and Systems 16 (1985), 265-275.

[3] P. P. Ming and L. Y. Ming, Fuzzy topology. II. Product and quotient spaces, J. Math. Anal. Appl. 77 (1980), 20-37.

[4] M. N. Mukherjee and S. P. Sinha, On some near-fuzzy continuous functions between fuzzy topological spaces, Fuzzy Sets and Systems 34 (1990), 245-254.

[5] J. H. Park, Generalizations of fuzzy continuous mappings between fuzzy topological spaces, *Dong-A University, Ph. D.*

[6] J. H. Park, J. R. Choi and B. Y. Lee, On fuzzy θ-continuous mappings, Fuzzy Sets and Systems 54 (1993), 107-113.

[7] S. Saha, Fuzzy δ-continuous mappings, J. Math. Anal. Appl. 126 (1987), 130-142.