A note on fuzzy continuous set-valued mappings.

K.Hur *, J.H.Ryou *, and Y.S.Ahn **

1. preliminaries

For a set X, let I^X denote set of all fuzzy sets, where $I=[0,1]$. Refer the concepts of union, intersection, complement, inclusion of two fuzzy sets to [6].

Now in order to deal with later sections, we list some definitions and results:

Definition 1.1[1]. Let $f: X\to Y$ be a mapping, $A\in I^X$ and $B\in I^Y$. Then:

(1) The *inverse image of* B *under* f, $f^{-1}(B)$ is a fuzzy set in X defined by for each $x\in X$,

$$[f^{-1}(B)](x) = B(f(x)) = (B\circ f)(x).$$

(2) The *image of* A *under* f, denoted by $f(A)$, is a fuzzy set in Y defined by for each $y\in Y$,

$$[f(A)](y) = \begin{cases} \sup_{y=f(x)} A(x) & \text{if } y\in f(X), \\ 0 & \text{if } y\notin f(X). \end{cases}$$

By the above definition,

$$f: I^X\to I^Y \text{ and } f^{-1}: I^Y\to I^X \text{ are mappings.}$$

Result 1.A[1,3]. Let $f: X\to Y$, $\{A_\alpha\}_{\alpha\in\Lambda}\subset I^X$, and $\{B_\alpha\}_{\alpha\in\Lambda}\subset I^Y$. Then:

(1) $f^{-1}(\bigcup_{\alpha\in\Lambda} B_\alpha) = \bigcup_{\alpha\in\Lambda} f^{-1}(B_\alpha)$, $f^{-1}(\bigcap_{\alpha\in\Lambda} B_\alpha) = \bigcap_{\alpha\in\Lambda} f^{-1}(B_\alpha)$.

(2) $f(\bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcup_{\alpha\in\Lambda} f(A_\alpha)$, $f(\bigcap_{\alpha\in\Lambda} A_\alpha)\subset \bigcap_{\alpha\in\Lambda} f(A_\alpha)$.

(3) $f(f^{-1}(B))\subset B$, $A\subset f^{-1}(f(A))$, where $A\in I^X$ and $B\in I^Y$.

Proposition 1.2. Let $f: X\to Y, A\in I^X$ and $B\in I^Y$. Then:

(1) $f(A)=\emptyset$ if and only if $A=\emptyset$. (2) $f(A)\cap B = f(A\cap f^{-1}(B))$.

Definition 1.3[4]. Let $A, B\in I^X$ and $x_\lambda\in F_p(X)$, where $F_p(X)$ denotes the set of all fuzzy points in X. Then :

(1) A is said to be *quasi-coincident with* B, denoted by AqB, if there exists an $x\in X$ such that $A(x) > B^c(x)$ or $A(x)+B(x) > 1$.

Also, we say that A and B are *quasi-coincident*(with each other) at x.

(2) x_λ is said to be *quasi-coincident with* A, denoted by $x_\lambda qA$, if $\lambda > A^c(x)$ or

$\lambda + A(x) > 1$.

Result 1.B[4]. Let $A, B \in I^X$ and $x_\lambda \in F_p(X)$. Then :

(1) $A \subset B$ if and only if $A \bar{q} B^c$. (2) $x_\lambda \in A$ if and only if $x_\lambda \bar{q} A^c$.

Definition 1.4[1]. A mapping $f: (X, \mathfrak{T}) \rightarrow (Y, \mathcal{U})$ is said to be *fuzzy continuous*(*F-continuous*, in short) if $f^{-1}(B) \in \mathfrak{T}$ for each $B \in \mathcal{U}$. The mapping f is called a *fuzzy homeomorphism*(*F-homeomorphism*, in short) if f is bijective, and both f and f^{-1} are F-continuous.

Result 1.C[5]. Let $f: (X, \mathfrak{T}) \rightarrow (Y, \mathcal{U})$ be a mapping. Then the following are equivalent :

(1) f is F-continuous.

(2) For each closed set B in Y, $f^{-1}(B)$ is closed in X.

(3) For each member V of a subbase $\mathcal{S}$ for $\mathcal{U}$, $f^{-1}(V) \in \mathfrak{T}$

(4) For each $x_\lambda \in F_p(X)$ and each neighborhood V of $f(x_\lambda)\ (= [f(x)]_\lambda)$, there exists a neighborhood U of x_λ such that $f(U) \subset V$.

(5) For each $x_\lambda \in F_p(X)$ and each q-neighborhood V of $f(x_\lambda)$, there exists a q-neighborhood U of x_λ such that $f(U) \subset V$.

(6) $f(\overline{A}) \subset \overline{f(A)}$ for each $A \in I^X$.

(7) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for each $B \in I^Y$.

From Result 1.C, we obtain the following result :

Proposition 1.5. Let $f: (X, \mathfrak{T}) \rightarrow (Y, \mathcal{U})$ be a mapping and $x_\lambda \in F_p(X)$. Then the following are equivalent :

(1) f is F-continuous at x_λ.

(2) $x_\lambda \in f^{-1}(B) \Rightarrow x_\lambda \in f^{-1}(B)$ for each $B \in I^Y$.

(3) $x_\lambda \in \overline{f^{-1}(B)} \Rightarrow x_\lambda \in f^{-1}(\overline{B})$ for each $B \in I^Y$.

Notation 1.6[2]. Let X be a fts and let $A \in I^X$. Then :

(1) $I_0^X = \{ E : E$ is a nonempty fuzzy closed set in $X \}$. (2) $I_0^A = \{ E \in I_0^X : E \subset A \}$.

Definition 1.7[2]. Let $(X, \mathfrak{T})$ be a fts. Then the *fuzzy Vietories topology* $\mathfrak{T}_v$ on I_0^X is generated by the collection of the forms $\langle U_1, \cdots, U_n \rangle_v$ with $U_1, \cdots, U_n$ fuzzy open sets in X, where $\langle U_1, \cdots, U_n \rangle_v = \{ E \in I_0^X : E \subset \bigcup_{i=1}^{n} U_i$ and $E\, q\, U_i$ for each $i = 1, \cdots n \}$.

The pair $(I_0^X, \mathfrak{T}_v)$ is called a *fuzzy hyperspace with fuzzy Vietories topology(fuzzy hyperspace*, in short).

Result 1.D[2]. Let $(X, \mathfrak{T})$ be a fts. Then :

(1) A is F-open in X if and only if I_0^A and $I_0^X - I_0^{A^c}$ are F-open in I_0^X.

(2) If A is F-closed in X, then I_0^A and $I_0^X - I_0^{A^c}$ are F-closed in I_0^X.

2. Definitions and fundamental property

For each $A \in I^X$, let I^A denote the set of all fuzzy sets in X contained in A. Hence $I^A = \{ E \in I^X : E \subset A \}$.

Definition 2.1. A mapping is said to be *fuzzy set-valued* if its values are fuzzy sets in a given set.

Hence, for instance, $f: I^X \to I^Y$ and $f^{-1}: I^Y \to I^X$ are fuzzy set-valued(See Definition 1.1).

Definition 2.2. Let $F_1, F_2 : Y \to I^X$ be fuzzy set-valued mappings. Then :

(1) $F_1 \subset F_2$ if and only if $F_1(y) \subset F_2(y)$ for each $y \in Y$.

(2) $F = F_1 \cup F_2$ if and only if $F(y) = F_1(y) \cup F_2(y)$ for each $y \in Y$.

(3) $F = F_1 \cap F_2$ if and only if $F(y) = F_1(y) \cap F_2(y)$ for each $y \in Y$.

Clearly, the set $(I^X)^Y$ can be considered as a complete distributive lattice.

Definition 2.3. Let $F: Y \to I^X$ be fuzzy set-valued and $A \in I^X$. Then the *inverse image of I^A under F*, denoted by $F^{-1}(I^A)$, is defined by

$$F^{-1}(I^A) = \{ y \in Y : F(y) \in I^A \} = \{ y \in Y : F(y) \subset A \}$$

It is clear that $Y - F^{-1}(I^A) = \{y \in Y : F(y) \notin I^A\} = \{y \in Y : f(y) q A^c\}$ by Result 1.B(1).

From Definition 2.2. and 2.3, we obtain easily the following result :

Theorem 2.4. Let $F_1, F_2 : Y \to I^X$ be fuzzy set-valued and $A \in I^X$. Then :

(1) If $F_1 \subset F_2$, then $F_2^{-1}(I^A) \subset F_1^{-1}(I^A)$.

(2) If $F = F_1 \cup F_2$, then $F^{-1}(I^A) = F_1^{-1}(I^A) \cap F_2^{-1}(I^A)$.

(2a) If $F_\alpha : Y \to I^X$ is fuzzy set-valued for each $\alpha \in \Lambda$, then

$$(\bigcup_{\alpha \in \Lambda} F_\alpha)^{-1}(I^A) = \bigcap_{\alpha \in \Lambda} F_\alpha^{-1}(I^A).$$

(3) If $F = F_1 \cap F_2$, then $F_1^{-1}(I^A) \cup F_2^{-1}(I^A) \subset F^{-1}(I^A)$.

(3a) If $F_\alpha : Y \to I^X$ is fuzzy set-valued for each $\alpha \in \Lambda$, then

$$\bigcup_{\alpha\in\Lambda} F_\alpha^{-1}(I^A) \subset (\bigcap_{\alpha\in\Lambda} F_\alpha)^{-1}(I^A).$$

3. F-continuity of fuzzy set-valued mappings

Theorem 3.1. Let Y be a fts, I_0^X a fuzzy hyperspace and $F: Y \to I_0^X$ a fuzzy set-valued mapping. Then the following are equivalent :

(1) F is F-continuous.

(2) For each fuzzy closed(resp. open) set A in X, $F^{-1}(I_0^A)$ is open(resp. closed) in Y.

(3) For each fuzzy closed(resp. open) set A in X, $Y - F^{-1}(I_0^{A^c})$is closed(resp. open) in Y.

Corollary 3.1. F is F-continuous at $y_0 \in Y$ if and only if both implications hold :

$$y_0 \in F^{-1}(I_0^G) \Rightarrow y_0 \in F^{-1}(I_0^G) \text{ whenever } G \text{ is a fuzzy open set in } X,$$

and

$$y_0 \in \overline{F^{-1}(I_0^K)} \Rightarrow y_0 \in F^{-1}(I_0^K) \text{ whenever } K \text{ is a fuzzy closed set in } X,$$

Theorem 3.2. Let $f: X \to Y$ be F-continuous. Then :

(1) $f^{-1}: I_0^Y \to I_0^X$ is F-continuous if and only if f is simultaneously F-closed and F-open.

(2) If f is F-closed, then $f: I_0^X \to I_0^Y$ is F-continuous.

Theorem 3.3. Let $F_1, F_2: Y \to I_0^X$ be fuzzy set-valued. If F_1 and F_2 are F-continuous, then $F_1 \cup F_2$ is F-continuous.

Remark 3.3. Theorem 3.3 can be stated in the following local form :
The union of two F-continuous mappings at y_λ is F-continuous at y_λ.

Corollary 3.3. The union $K \cup L$, considered as a mapping of $I_0^X \times I_0^X$ onto I_0^X, is F-continuous.

References

[1] C.L.Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24(1968) 182-190.

[2] K.Hur, J.R.Moon and J.H.Ryou, A note on fuzzy Vietories topology, Journal of Fuzzy Logic and Intelligent Systems, Vol. 10 No.2(2000) 129-132.

[3] C.V. Negoite and D.A. Ralescu, Applications of Fuzzy Sets to Systems Analysis, Birkhäuser Verlag. Basel, 1975.

[4] Pu Pao-Ming and LiuYing-Ming, Fuzzy topology I, J.Math.Anal.Appl. 76(1980) 571-599.

[5] ______________ , Fuzzy topology II, J.Math.Anal.Appl. 77(1980) 20-37.

[6] L.A.Zadeh, Fuzzy sets, Inform and Control 8(1965)338-353.