Bayesian Analysis for Burr-Type X Strength-Stress Model

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Abstract

In this paper, we develop noninformative priors that are used for estimating the reliability of stress-strength system under the Burr-type X distribution. A class of priors is found by matching the coverage probabilities of one-sided Bayesian credible interval with the corresponding frequentist coverage probabilities. It turns out that the reference prior is a first order matching prior. The propriety of posterior under matching prior is provided. The frequentist coverage probabilities are given for small samples.

1. INTRODUCTION

Consider the following stress-strength system, where Y is the strength of a system subject to stress X. The system fails at any moment the applied stress is greater than its strength. Then reliability of the system is given by

$$w_1 = P(X < Y). \quad (1)$$

This model was first considered by Birnbaum (1956), and has since then found applications in many areas, especially in structural and aircraft industries. Basu (1985) and Johnson (1988) contain reviews of many results primarily related to frequentist inference for stress-strength models, although Johnson (1988) contains a small Bayesian result.

The present paper focuses exclusively on Bayesian inference for w_1. The emphasis is on noninformative priors. Although subjective Bayesian are often critical of such priors, these priors have clear pragmatic appeal especially when prior information is vague in nature.

The most frequently used noninformative prior is Jeffrey's (1961) prior, which is proportional to the positive square root of the determinant of the Fisher information matrix. In the one-parameter case, Welch and Peers (1963) proved that a one-sided credible interval from Jeffrey's prior matches the corresponding frequentist coverage probability up to $o(n^{-1/2})$.

In spite of its success in one-parameter problems, Jeffrey's prior frequently runs into serious difficulties in the presence of nuisance parameters. As an alternative, we use the method of Welch and Peers (1963), Peers (1965) and Stein (1985) to find a prior which requires the frequentist coverage probability of the posterior region of a real-valued parametric function to match the normal level with a remainder of $o(n^{-1/2})$, where n is the sample size. Tibshirani (1989) reconsidered the case when the real valued parameter of interest is orthogonal to the nuisance parameter vector. These priors, as usually referred to as 'first order' matching priors, were further studied in Datta and Ghosh (1995, 1996).

On the other hand, Ghosh and Mukerjee (1992), and Berger and Bernardo (1989,1992) extended Bernardo's (1979) reference prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion described earlier.

Thompson and Basu (1993) derived reference priors when the stress and strength are both exponentially distributed. It turns out that in such cases, the reference priors agree with Jeffrey's prior. Lee, Sun and Basu (1997) derived matching priors and Ghosh and Yang (1996) derived reference priors when the stress and strength are both normal distributed. Sun, Ghosh and Basu (1998) derived Jeffrey's prior, reference priors and matching priors when the stress and strength are both Weibull distributed. It turns out that none of the Jeffrey's prior and the reference priors is a matching prior. Their study shows that the matching prior performs better than Jeffrey's prior and reference priors in meeting the target coverage probabilities.

Ahmad, Fakhry and Jaheen (1997) studied the empirical Bayes estimation of $P(X<Y)$ when X and Y are independent Burr-type X random variables.

The present paper derive matching priors as well reference priors for Burr-type X stress-strength models when w_1 is the parameter of interest. Surprisingly, reference priors, first order matching prior and Jeffrey's prior are agree.

The outline of the remaining sections is as follows. In Section 2, we derive Fisher information matrices under original parameterization and reparametrization. Then we derive two possible reference priors and provide a general class of matching priors of which a particularly simple one is recommended. The propriety of posterior under this prior is given. Also the marginal density of w_1 under this prior is given. In Section 3, simulated frequentist coverage probabilities under this prior are given for small samples.

2. REFERENCE AND MATCHING PRIORS

We denote the Burr-type X distribution with probability density function(pdf)

$$f(x;\theta)=2\theta xe^{-x^2}(1-e^{-x^2})^{\theta-1},\ x>0,\ \theta>0, \quad (2)$$

as Burr-type X (θ). Suppose that $X_1,\cdots,X_m$ are independent random samples from Burr-type X (η_1), and independently, $Y_1,\cdots,Y_n$ are independent random samples from Burr-type X (η_2). The log-likelihood function of (η_1,η_2) is

$$\begin{aligned} l(\eta_1,\eta_2) &\propto m\log(\eta_1)+n\log(\eta_2) \\ &-\eta_1\sum_{i=1}^{m}\log(1-e^{-x_i^2})-\eta_2\sum_{j=1}^{n}\log(1-e^{-y_j^2}). \end{aligned} \quad (3)$$

By a simple computation, we get the following Lemma.

Lemma 1. The Fisher information matrix of (η_1, η_2) is given by

$$\begin{pmatrix} \frac{m}{\eta_1^2} & 0 \\ 0 & \frac{n}{\eta_2^2} \end{pmatrix}.$$

Then Jeffrey's prior is

$$\pi_J(\eta_1, \eta_2) \propto \frac{1}{\eta_1 \eta_2}. \quad (4)$$

Under the Burr-type X Stress-Strength model, the parameter of interest defined in (1) is

$$w_1 = \frac{\eta_2}{\eta_1 + \eta_2}. \quad (5)$$

To apply the reference prior approach, one nuisance parameter is needed. We consider $w_2 = \eta_1 + \eta_2$. The transformation from (η_1, η_2) to (w_1, w_2) is one to one with the inverse transformation

$$\eta_1 = (1 - w_1) w_2, \quad \eta_2 = w_1 w_2. \quad (6)$$

The following Lemma provide the Fisher information matrix and its determinant.

Lemma 2. (i) The Fisher information matrix of (w_1, w_2) has the following form:

$$J = \begin{pmatrix} \frac{n}{w_1^2} + \frac{m}{(1 - w_1^2)} & \frac{n}{w_1 w_2} - \frac{m}{w_2(1 - w_1)} \\ \frac{n}{w_1 w_2} - \frac{m}{w_2(1 - w_1)} & \frac{n+m}{w_2^2} \end{pmatrix}.$$

(ii) The determinant of J is

$$\text{Det}(J) = \frac{nm}{w_1^2 (1 - w_1)^2 w_2^2}.$$

Bernardo (1979) suggested determining reference priors in two steps. This method has been modified by Berger and Bernardo (1989, 1992) who provided a general algorithm to cover situations with multiple group ordered in term of inferential importance. It is possible to have many different ways to order the parameters in order to obtain many different reference priors. For simplicity, the notation $\{w_1, w_2\}$ will be used to represent the case where are two groups, with w_1 being the most important. In a discussion of Ghosh and Mukerjee (1992), Berger (1992) suggested that switching the role of the parameters of interest and nuisance parameters sometimes gives a reasonable reference prior. The corresponding prior is called the reverse reference prior. So, we also consider the grouping order $\{w_2, w_1\}$.

We prove only one case, the grouping order $\{w_1, w_2\}$. Other is similar. We will use the notation of Berger and Bernardo (1992). For the proof below, we choose $\Omega_k = [a_{1k}, b_{1k}] \times [a_{2k}, b_{2k}]$, such that

$$a_{1k}, a_{2k} \to 0, \; b_{1k} \to 1, \; b_{2k} \to \infty, \text{ as } k \to \infty.$$

Theorem 1. (i) The reference prior for $\{w_1, w_2\}$ is $\pi_R(w_1, w_2) = \frac{1}{w_1(1 - w_1) w_2}$. (ii) The reference prior for $\{w_2, w_1\}$ is

$$\pi_R(w_1, w_2) = \frac{1}{w_1(1 - w_1) w_2}.$$

Proof. We need to find h_1 and h_2. We show that

$$h_1 = \frac{\det(J)}{\det(J_{22})} = \frac{nm}{(n+m) w_1^2 (1 - w_1)^2},$$

$$h_2 = \det(J_{22}) = \frac{(m+n)}{w_2^2}.$$

Then the conditional prior of w_2 given w_1 is

$$\xi_2^k(w_2 \mid w_1) = \frac{\sqrt{h_2} I_{[a_{2k}, b_{2k}]}(w_2)}{\int_{a_{2k}}^{b_{2k}} \sqrt{h_2} dw_2} = \frac{I_{[a_{2k}, b_{2k}]}(w_2)}{w_2 \log(b_{2k}/a_{2k})},$$

and the marginal prior of w_1 is

$$\xi_1^k(w_1) = \exp\{\frac{1}{2} E^k[\log(h_1) \mid w_1]\} I_{[a_{1k}, b_{1k}]}(w_1) \propto \frac{1}{w_1(1-w_1)} I_{[a_{1k}, b_{1k}]}(w_1)$$

Therefore, the reference prior for $\{w_1, w_2\}$ is

$$\pi_R(w_1, w_2) = \lim_{k \to \infty} \frac{\xi_1^k(w_1)\xi_2^k(w_2 \mid w_1)}{\xi_1^k(0.5)\xi_2^k(1 \mid 0.5)},$$

which is the desired result.

By transformation from (w_1, w_2) to (η_1, η_2), we obtain the reference prior for (η_1, η_2)as follow:

$$\pi_R(\eta_1, \eta_2) = \frac{1}{\eta_1 \eta_2}. \qquad (7)$$

It is of interest whether we can find matching prior within reference priors. There are many examples where one of the reference priors is a matching prior. (See Datta and Ghosh (1995), Sun and Ye (1995, 1996). We will derive general forms of matching priors for w_1 following Datta and Ghosh (1995).

Theorem 2. The general class of matching priors for $\{\eta_1, \eta_2\}$ is of the form

$$\pi_M(\eta_1, \eta_2) = \frac{1}{\eta_1 \eta_2} g(\frac{1}{\eta_1^m \eta_2^n}). \qquad (8)$$

Proof. The gradient vector is given by

$$\nabla w_1 = (\frac{\partial}{\partial \eta_1} w_1, \frac{\partial}{\partial \eta_2} w_1)^T = \frac{1}{(\eta_1 + \eta_2)^2}(-\eta_2, \eta_1)^T.$$

Define $\xi = (\xi_1, \xi_2)^T = \Sigma^{-1} \nabla w_1 / \sqrt{(\nabla w_1)^T \Sigma^{-1} \nabla w_1}$. Following Datta and Ghosh (1995), if the posterior probability of a one-sided credibility interval for w_1 and its frequentist probability agree up to $o(n^{-1/2})$, the prior π should satisfy the equation:

$$\frac{\partial}{\partial \eta_1}(\xi_1 \pi) + \frac{\partial}{\partial \eta_2}(\xi_2 \pi) = 0.$$

Then the equation simplify to

$$\frac{\partial}{\partial \eta_2} \frac{\eta_2}{n} \pi(\eta_1, \eta_2) - \frac{\partial}{\partial \eta_1} \frac{\eta_1}{m} \pi(\eta_1, \eta_2) = 0. \qquad (9)$$

A general class of the solutions to (9) is of the form

$$\pi_M(\eta_1, \eta_2) = \frac{1}{\eta_1 \eta_2} g(\frac{1}{\eta_1^m \eta_2^n}).$$

Clearly there are infinitely many matching priors for w_1 up $o(n^{-1/2})$. A solution is to use the simplest one among all the matching priors, where $g \equiv 1$. The corresponding prior for (w_1, w_2) is then

$$\pi_M(\eta_1, \eta_2) = \frac{1}{\eta_1 \eta_2}. \qquad (10)$$

Thus it turn out that the Jeffrey's prior, the reference priors and the first order matching prior for (η_1, η_2) are the same. Therefore we denoted by

$$\pi(\eta_1, \eta_2) \equiv \pi_J(\eta_1, \eta_2) = \pi_R(\eta_1, \eta_2) = \pi_M(\eta_1, \eta_2). \qquad (11)$$

We now provide that the posterior is proper under the first order matching prior.

Theorem 3. The posterior distribution of (η_1, η_2) under the prior π, (11), is proper.

Proof.

$$\int_0^\infty \int_0^\infty L(\eta_1, \eta_2) \frac{1}{\eta_1 \eta_2} d\eta_1 d\eta_2$$
$$= 2^{m+n} \prod_{i=1}^{m} \frac{x_i}{(1-\exp(-x_i^2))} \prod_{j=1}^{n} \frac{y_j}{(1-\exp(-y_j^2))} \exp(-\sum_{i=1}^{m} x_i^2 - \sum_{j=1}^{n} y_j^2)$$
$$\times \int_0^\infty \int_0^\infty \eta_1^{m-1} \eta_2^{n-1} [\prod_{i=1}^{m}(1-\exp(-x_i^2))]^{\eta_1} [\prod_{j=1}^{n}(1-\exp(-y_j^2))]^{\eta_2} d\eta_1 d\eta_2$$
$$= 2^{m+n} \prod_{i=1}^{m} \frac{x_i}{(1-\exp(-x_i^2))} \prod_{j=1}^{n} \frac{y_j}{(1-\exp(-y_j^2))} \exp(-\sum_{i=1}^{m} x_i^2 - \sum_{j=1}^{n} y_j^2)$$
$$\times \frac{\Gamma(m)}{(\sum_{i=1}^{m} \log[1-\exp(-x_i^2)])^m} \frac{\Gamma(n)}{(\sum_{j=1}^{n} \log[1-\exp(-y_j^2)])^n}$$
$$< \infty .$$

Next, we provide the marginal density of w_1 under this prior.

Theorem 4. Under the prior π, (11), the marginal posterior density of $w_1 = \eta_2 / (\eta_1 + \eta_2)$, is given by

$$\pi(w_1 \mid \boldsymbol{X}, \boldsymbol{Y}) \propto w_1^{n-1}(1-w_1)^{m-1}\left(\frac{1}{h(w_1)}\right)^{m+n}, \quad (12)$$

where

$$h(w_1) = \log[\prod_{i=1}^{m}(1-\exp(-x_i^2))] + w_1 \log[\frac{\prod_{j=1}^{n}(1-\exp(-y_j^2))}{\prod_{i=1}^{m}(1-\exp(-x_i^2))}].$$

The normalizing constant for the marginal density of w_1 requires a one dimensional integration. Therefore we have the marginal posterior density of w_1, it is easy to compute the marginal moment of w_1.

3. SMALL SAMPLE STUDY

We investigate in finding the credible interval for w_1 when m and n are small under the matching prior π. That is to say, the frequentist coverage of a $(1-\alpha) th$ posterior quantile should be close to $(1-\alpha)$. This is done numerically. Table 1 gives numerical values of the frequentist coverage probabilites of 0 to 0.05 (0 to 0.95) posterior quantiles, for the matching prior. The computation of these numerical values is based on the following algorithm for any fixed true (η_1, η_2) and any prespecified probability value α. Here α is 0.05 (0.95). Let $w_1^{\pi}(\alpha \mid \boldsymbol{X}, \boldsymbol{Y})$ be the posterior α-quantile of w_1 given $(\boldsymbol{X}, \boldsymbol{Y})$. That is to say, $F(w_1^{\pi}(w_1 \mid \boldsymbol{X}, \boldsymbol{Y}) \mid \boldsymbol{X}, \boldsymbol{Y}) = \alpha$, where $F(\cdot \mid \boldsymbol{X}, \boldsymbol{Y})$ is the marginal posterior distribution of w_1. Then the frequentist coverage probability of this one sided credible interval of w_1 is

$$P_{(\eta_1, \eta_2)}(\alpha; w_1) = P_{(\eta_1, \eta_2)}(0 < w_1 \le w_1^{\pi}(\alpha \mid \boldsymbol{X}, \boldsymbol{Y})) \quad (13)$$

The estimated $P_{(\eta_1, \eta_2)}(\alpha; w_1)$ when $\alpha = 0.05(0.95)$ is shown in Table 1. In particular,

Table 1. Frequentist Coverage Probability of 0.05 (0.95) Posterior Quantiles of w_1

η_1	η_2	m	n	π	
				0.05	0.95
1	1	2	2	0.0508	0.9472
1	1	2	3	0.0501	0.9486
1	1	5	5	0.0483	0.9465
1	1	10	10	0.0474	0.9499
3	2	2	2	0.0468	0.9529
3	2	2	3	0.0519	0.9527
3	2	5	5	0.0491	0.9517
3	2	10	10	0.0524	0.9493
5	5	2	2	0.0486	0.9509
5	5	2	3	0.0454	0.9525
5	5	5	5	0.0492	0.9478
5	5	10	10	0.0516	0.9526

for fixed (η_1, η_2, m, n), we take 10,000 independent random samples of m stresses $\boldsymbol{X}=(X_1,\cdots,X_m)$ from Burr-type X (η_1) and n strengths $\boldsymbol{Y}=(Y_1,\cdots,Y_n)$ from Burr-type X (η_2). Note that under the prior π, for fixed $\boldsymbol{X}$ and $\boldsymbol{Y}$, $w \leq w_1^\pi(\alpha \mid \boldsymbol{X}, \boldsymbol{Y})$ if and only if $F(w_1^\pi(\alpha \mid \boldsymbol{X}, \boldsymbol{Y}) \mid \boldsymbol{X}, \boldsymbol{Y}) \leq \alpha$. Under the prior π, $P_{(\eta_1,\eta_2)}(\alpha; w_1)$ can be estimated by the relative frequency of $F(w_1^\pi \mid \boldsymbol{X}, \boldsymbol{Y}) \leq \alpha$. For the cases presented in Table 1, we see that the first order matching prior π_M meet very well the target coverage probabilities.

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